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Anomalous monopoles of an interacting boson system

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ABSTRACT

Two anomalous monopoles, one of line shape and the other of disk shape, are found to exist in the semiclassical theory of a two-mode interacting boson system. This is in stark contrast with the quantum theory of this system, where only point-like monopoles exist. We show that these two anomalous monopoles have different origins. The line-shaped monopole is formed from the merging of a series of point-like monopoles while the disk-shaped monopole is the result of the collapsing or bundling of field lines of Berry curvature due to the existence of the influence of the interaction between bosons. The relation of these two anomalous monopoles with the famed von Neumann–Wigner theorem is discussed.

1. Introduction

Magnetic monopole was first suggested by Dirac as a quantization condition for electric charge [1]. Although its existence as a fundamental particle has not been confirmed by experiment, the monopole has fascinated physicists ever since [2]. Interestingly, monopoles can arise in a very different context as degeneracy or diabolical points of energy levels in a parameter space [3–7]. This is because degeneracy point resembles the magnetic monopole in two aspects: i) it generates "magnetic" field (i.e., Berry curvature) in the parameter space; ii) its charge is quantized as Chern number and is a multiple of 2π [5]. The monopole as degeneracy point is found to be crucial to understanding some physical effects. For example, the monopole in the Brillouin zone is found to play a pivotal role in the anomalous Hall effect [8].

In this work we study the monopoles of a two-mode interacting boson system, which depends on three external parameters. When the number of bosons N in the system is finite, this system can be described by a second-quantized model. In the large N limit, $N \to \infty$, this boson system can be described alternatively by a mean-field theory [9,10]. Since the large N limit is also a semiclassical limit [11], this mean-field model can be regarded as the semiclassical theory of the second-quantized model. We examine the monopoles of this boson system with the second-quantized model for finite N and the mean-field model for the large N limit, $N \to \infty$. Our focus is on the ground state and the highest eigenstate.

Our analysis with the second-quantized model for the case of finite N shows that the monopoles are point-like objects in the three-dimensional parameter space. Specifically, for the highest eigenstate, the system is doubly degenerate at N evenly-spaced points along a line of fixed length in the parameter space. For the ground state, there is only one isolated degeneracy point. This means that we have only one point-like monopole for the ground state while we have a series of point-like monopoles for the highest eigenstates. As we shall show in the text, the monopoles being point-like in the parameter space is the result of the famed von Neumann–Wigner (vNW) theorem [12]. This theorem states that one has to change three parameters in a Hermitian matrix to achieve double degeneracy and $m^2 - 1$ parameters to obtain m-fold degeneracy.

However, at large N limit, the monopoles for these two eigenstates are no longer points. Our computation with the mean-field model shows that the highest eigenstate is degenerate on a line and the ground state is degenerate on a two-dimensional disk. It is equivalent to say that the monopoles have become anomalous with one being line-shaped and the other being disk-shaped (see Fig. 1). The appearance of these two anomalous monopoles indicates that the vNW theorem may become non-applicable in the semiclassical limit, $N \to \infty$.

These two anomalous monopoles have very distinct origins. As noted above, in the case of finite N, the highest eigenstate has N point-like monopoles on a line of fixed length. The line-shaped monopole is formed naturally from the merging of these

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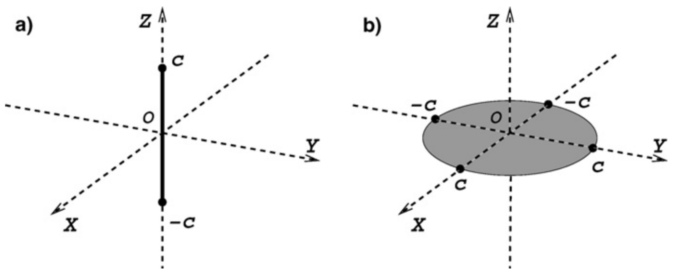

Fig. 1. Anomalous monopoles in the semiclassical limit. (a) Line-shaped monopole for the highest eigenstate; (b) Disk-shaped monopole for the ground state.

N monopoles in the limit of $N \to \infty$. In contrast, the ground state has only one monopole for any finite N. The appearance of the disk-shaped monopole in the semiclassical limit is very sudden and even mysterious at first encounter. After detailed study, we find that the origin of the disk-shaped monopole is found to be related to Berry curvature [4]. As Berry curvature can be regarded as the "magnetic field" generated by the monopole in the parameter space [13], we borrow the basic tool in electromagnetism, field lines, to illustrate the Berry curvature for the ground state. Our computation with the second-quantized model shows that the field lines of Berry curvature emanating from the monopole are curved towards a disk due to the existence of the interaction between bosons. And the curving of the field lines becomes more severe as N becomes larger. Eventually, at large N limit, the field lines completely collapse and bundle into a disk, which is exactly the anomalous monopole found in the mean-field model. Our further analysis shows that the charge is not uniformly distributed in the monopole disk while its total charge divided by N is kept at 2π, the Chern number.

Berry curvature is also computed for this system with the mean-field model and the results are compared to the ones obtained with the second-quantized model. The matching becomes better as N increases, confirming a semiclassical relation between quantum and classical two-forms established by Berry [14]. However, on the monopole disk, the Berry curvature differs significantly between the semiclassical result and quantum result even in the large N limit. This shows that the anomalous monopole of disk shape also indicates a breakdown of the Berry's semiclassical relation.

2. Two-mode interacting boson system

The two-mode interacting system of N bosons is described by the following second-quantized Hamiltonian

$$\hat{H}_N = \frac{X}{2}\left(\hat{a}^\dagger\hat{b} + \hat{a}\hat{b}^\dagger\right) + \frac{iY}{2}\left(\hat{a}\hat{b}^\dagger - \hat{a}^\dagger\hat{b}\right) + \frac{Z}{2}\left(\hat{a}^\dagger\hat{a} - \hat{b}^\dagger\hat{b}\right) - \frac{\lambda}{4V}\left(\hat{a}^\dagger\hat{a} - \hat{b}^\dagger\hat{b}\right)^2, \tag{1}$$

where $\hat{a}^\dagger$, $\hat{a}$ and $\hat{b}^\dagger$, $\hat{b}$ are bosonic operators for two different quantum states, respectively, $\lambda > 0$ is the interaction strength between bosons, and V is the volume of the system. Among the three parameters, X and Y are couplings between these two modes and Z is the energy difference between these two modes. This Hamiltonian has been widely used in modeling the Bose–Einstein condensate in a double-well potential and in other situations where only two modes are important [9,10]. It also belongs to a class of Hamiltonians studied in Refs. [6,7] for single molecular magnet if we introduce $\hat{J}_x = (\hat{a}^\dagger\hat{b} + \hat{b}^\dagger\hat{a})/2$, $\hat{J}_y = i(ab^+ - a^\dagger b)/2$, and $\hat{J}_z = (\hat{a}^\dagger\hat{a} - \hat{b}^\dagger\hat{b})/2$. For molecular magnets, X, Y, and Z represent three components of a magnetic field.

At large N limit, this boson system becomes "classical" [11] and can be described by the following mean-field (or semiclassical) Hamiltonian [9,10],

$$H_s = \lim_{N\to\infty}\frac{\hat{H}_N}{N} = \frac{X}{2}\left(a^*b + ab^*\right) + \frac{iY}{2}\left(ab^* - a^*b\right) + \frac{Z}{2}\left(|a|^2 - |b|^2\right) - \frac{c}{4}\left(|a|^2 - |b|^2\right)^2, \tag{2}$$

where a and b are complex amplitudes for the system in the two quantum modes. $c = N\lambda/V$. The normalization is one, i.e., $|a|^2 + |b|^2 = 1$. This kind of nonlinear Hamiltonian also appears in photoassociation systems [15]. For simplicity and without loss of essence, we focus on the ground state and the highest eigenstate of this system. Note that large N limit is always taken by keeping N/V constant.

3. Monopoles for the highest eigenstate

We consider first the highest eigenstate. If the system has N bosons, its highest eigenstate is doubly degenerate at N evenly-spaced points along a line of fixed length. These N degenerate points are located at $\{X = 0, Y = 0, Z/c = m/N\}$ with $m = -N+1, -N+3, \ldots, N-3, N-1$. At each of these points, the two degenerate eigenstates are $|(N+m+1)/2, (N-m-1)/2\rangle$ and $|(N+m-1)/2, (N-m+1)/2\rangle$.

As the number of boson N increases, these N monopoles get closer. Consequently, it is natural to expect that the system be degenerate on the whole line defined by $X = Y = 0$ and $-c < Z < c$ at large N limit. This expectation is confirmed by our computation with the mean-field model in Eq. (2): the mean-field highest eigenstate is indeed degenerate on the line. The mean-field highest eigenstate is given by

$$|\phi_h\rangle \equiv \begin{pmatrix} a \\ b \end{pmatrix} = \begin{pmatrix} e^{i\varphi}\sqrt{\frac{c+Z}{2c}} \\ \sqrt{\frac{c-Z}{2c}} \end{pmatrix}. \tag{3}$$

In the above, the phase φ is arbitrary, indicating degeneracy.

4. Monopoles for the ground state

We now turn to the ground state. When N is finite, the ground state is doubly degenerate at an isolated point at $X = Y = Z = 0$. The two degenerate ground states are $|N, 0\rangle$ and $|0, N\rangle$. So, the monopole for the ground state is a single point for any finite N. It appears to suggest that this monopole remain a point even in the large N limit. However, our computation with the mean-field model shows otherwise.

With some algebra, the ground state of the mean field model in Eq. (2) is found to be given by

$$|\phi_g\rangle \equiv \begin{pmatrix} a \\ b \end{pmatrix} = \begin{pmatrix} \sqrt{\frac{1-p}{2}} \\ -\sqrt{\frac{1+p}{2}}\frac{X+iY}{\sqrt{X^2+Y^2}} \end{pmatrix}, \tag{4}$$

where p is the solution of the following equation,

$$p\sqrt{X^2+Y^2} = (Z + cp)\sqrt{1-p^2}. \tag{5}$$

This equation has one real root when $X^2 + Y^2 \geqslant c^2$. When $X^2 + Y^2 < c^2$, it has three real roots when $|Z| < [c^{2/3} - (X^2+Y^2)^{1/3}]^{3/2}$. In particular, when $Z = 0$, two of the three real roots given by $p = \pm\sqrt{1 - (X^2+Y^2)/c^2}$ have the same energy and are for the ground states. This means that in the semiclassical description of the system, the ground state is degenerate on the disk given by

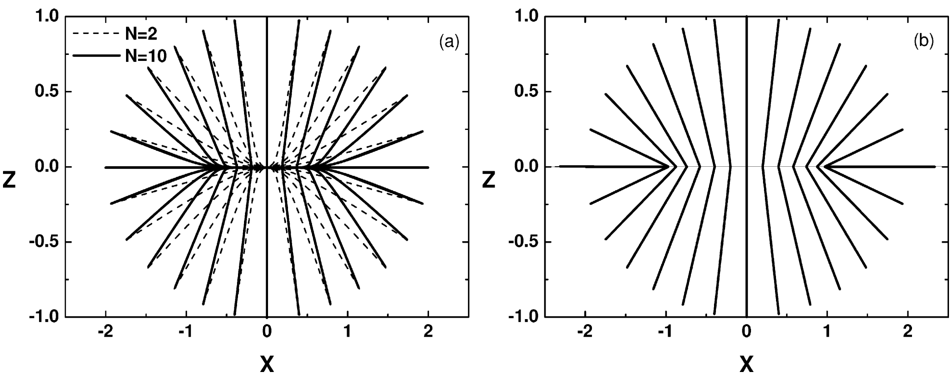

Fig. 2. Field lines of Berry curvature of the monopole at $X = Y = Z = 0$ for the ground state. (a) Dashed lines are for $N = 2$ and solid lines are for $N = 10$; (b) large N limit result obtained with the mean-field Hamiltonian. Due to the symmetry around Z-axis, the Y component is omitted. $c = 1$. The lines in (b) are not straight as they appear.

$X^2 + Y^2 < c^2$ and $Z = 0$. In other words, the whole disk is a monopole (see Fig. 1(b)).

This anomalous disk-shaped monopole is very surprising. There appears no indication of the emergence of the disk-shaped monopole at any finite N, where the monopole is always a single point at $X = Y = Z = 0$. This asks for further investigation and prompts us to look into Berry curvature, the "magnetic" field generated by degeneracy points in the parameter space. As one usually uses field lines to illustrate a magnetic field in electromagnetism, we have computed numerically the field lines for "magnetic field" $\mathbf{B}_N$ (or Berry curvature) of the monopole at $X = Y = Z = 0$ and plotted them in Fig. 2(a). For clarity, only the results for $N = 2$ and $N = 10$ are shown. Nevertheless, an interesting trend is clearly demonstrated: the field lines are curved towards the monopole disk defined by $\sqrt{X^2 + Y^2} = c$ and $Z = 0$; the curving gets stronger as N increases. In fact, our numerical results show that the field lines will collapse and bundle (or converge) into the disk when N approaches infinity. These results illustrate that the origin of the disk-shaped monopole is the collapsing (or converging) of field lines in the semiclassical limit, $N \to \infty$. This is in agreement with our common knowledge that a magnetic monopole (or an electric charge) in electromagnetism can be viewed as the converging point or the emitting source of field lines.

Let us examine this disk-shaped anomalous monopole in detail. Although the semiclassical Hamiltonian H_s is nonlinear, the Berry curvature $\mathcal{B}$ of this monopole can be computed as in a linear system [16]. That is to compute the curl of the vector potential $\mathbf{A} = \langle \phi_g | \nabla | \phi_g \rangle$ with $|\phi_g\rangle$ given in Eq. (4). The Berry curvature $\mathcal{B}$ is found to be

$$\mathcal{B} = \frac{p^3}{2(cp + Z)^2 (cp^3 + Z)} (\mathbf{R} + cp\hat{z}), \tag{6}$$

where $\mathbf{R} = \{X, Y, Z\}$ and $\hat{z}$ is the unit vector for Z direction. This result is plotted as field lines in Fig. 2(b). It is apparent that these semiclassical field lines away from the monopole disk are very similar to the field lines obtained with the second-quantized model. Note that $\mathcal{B}$ has two different values on the monopole disk due to the double degeneracy of the ground state. By integrating $\mathcal{B}$ over a closed surface around a small area in the disk, we find that the "magnetic" charge is not uniformly distributed over the disk. The charge distribution is

$$\rho = \frac{1}{c\sqrt{c^2 - (X^2 + Y^2)}}. \tag{7}$$

The integration of this charge density over the whole disk gives us a Chern number of 2π. So, although the monopole has changed from a point suddenly to a disk as the semiclassical limit is approached, the total charge does not change. Note that the total charge for the monopole in the second-quantized model is $2N\pi$.

Berry [14] once established a semiclassical relation between Berry phase [4] and Hannay's angle [17,18]. This semiclassical relation basically says that the two-forms, respectively, for Berry phase and Hannay's angle (the two-form for Berry phase is Berry curvature) are the same in the semiclassical limit $\hbar \to 0$. This semiclassical relation should hold in this interacting boson system. We define two pairs of conjugate variables, $p_a = \sqrt{i\hbar} a^*$, $q_a = \sqrt{i\hbar} a$ and $p_b = \sqrt{i\hbar} b^*$, $q_b = \sqrt{i\hbar} b$ for the semiclassical Hamiltonian (2) [19]. The quantization is realized with the following commutators,

$$[\hat{q}_a, \hat{p}_a] = [\hat{q}_b, \hat{p}_b] = i\hbar/N. \tag{8}$$

One can obtain the second-quantized Hamiltonian (1) with the following substitution $\hat{a} = \sqrt{N/i\hbar}\hat{q}_a$, $\hat{a}^\dagger = \sqrt{N/i\hbar}\hat{p}_a$ and $\hat{b} = \sqrt{N/i\hbar}\hat{q}_b$, $\hat{b}^\dagger = \sqrt{N/i\hbar}\hat{p}_b$. These commutators show that the semiclassical limit $\hbar \to 0$ is equivalent to $N \to \infty$ for this particular system. As a result, the semiclassical relation established by Berry [14] for this boson system is

$$\lim_{N\to\infty} \delta\mathbf{B} = \lim_{N\to\infty} \left(\frac{\mathbf{B}_N}{N} - \mathcal{B} \right) = 0. \tag{9}$$

Our numerical results show that the relation (9) indeed holds almost everywhere in the parameter space except on the monopole disk. On the disk, the semiclassical Berry curvature $\mathcal{B}$ has a nonzero $\hat{z}$ component while the quantum $\mathbf{B}_N$ always points radially in the $Z = 0$ plane. Furthermore, the quantum Berry curvature $\mathbf{B}_N$ diverges exponentially with N on the monopole disk while the in-plane component of the semiclassical $\mathcal{B}$ does not. We define $d = |\delta\mathbf{B}^l|$, where the superscript l denotes the component of the vector parallel to the XY plane. The difference d is plotted in Fig. 3, where we see the difference d increases exponentially with N. This diverging difference shows that the semiclassical relation in Eq. (9) is broken. Therefore, the disk-shaped monopole also signifies the breakdown of the Berry's semiclassical relation between quantum and classical two-forms [14].

5. Implications for the von Neumann–Wigner theorem

In this section, we discuss the implications of our above results for the famed von Neumann–Wigner (vNW) theorem [12], which deals with the problem of how many parameters need to be changed to achieve certain degeneracy in Hermitian matrices. We limit our discussion here to double degeneracy since it is the focus of this Letter. Consider a $n \times n$ Hermitian matrix V. It has n^2 free parameters that span a n^2-dimensional manifold M. This manifold has a submanifold M_s on which the Hermitian matrix V has at least two identical eigenvalues. The vNW theorem states that the dimension (or co-dimension) of the submanifold M_s is $n^2 - 3$ (or 3). What does this theorem say about a quantum system whose

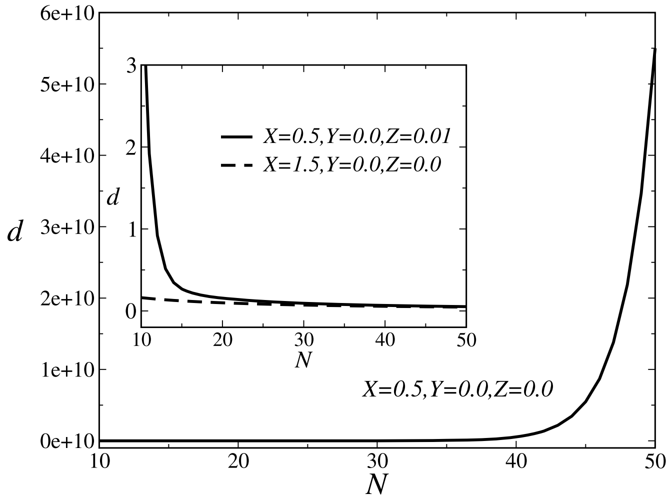

Fig. 3. The difference $d = |\delta \mathbf{B}^l|$ as a function of the number of bosons N at a point on the monopole disk. The inset shows the results for points away from the disk. $c = 1$.

Hamiltonian $\hat{H}(X_1, X_2, X_3)$ depends on three independent external parameters? How often does the system become degenerate as X_1, X_2, X_3 vary?

Let us consider a quantum system which has a n-dimensional Hilbert space. In this case, its Hamiltonian can be represented by a $n \times n$ Hermitian matrix V with a set of orthonormal basis. In this way, the Hamiltonian $\hat{H}(X_1, X_2, X_3)$ is a map from the three-dimensional space spanned by X_1, X_2, X_3 to the n^2-dimensional manifold M for Hermitian matrices. The "independence" between the three external parameters is to be guaranteed by requiring that the map be injective. In this way, we have excluded non-interesting cases such as

$$V = \begin{pmatrix} X_1 + X_2 + X_3 & (X_1 + X_2 + X_3)^2 \\ (X_1 + X_2 + X_3)^2 & -(X_1 + X_2 + X_3) \end{pmatrix}, \tag{10}$$

which maps a three-dimensional object into a one-dimensional object. To ask how often the system becomes degenerate is now equivalent to ask how the submanifold $\hat{H}(X_1, X_2, X_3)$ intersects with the submanifold M_s for the double degeneracy. It is clear that if there is intersection, the dimension of this intersection can be zero (points), one (lines), two (surfaces), and three. For the Hamiltonian considered in this work, we have zero-dimensional intersections (i.e., points). If the vNW theorem were different, say, it claimed that the co-dimension of M_s is two, then these point-like intersections (or monopoles) would be impossible. Therefore, we can say that these point-like monopoles for $\hat{H}_N$ are the results partially due to the vNW theorem. That they become either a line or a disk at large N limit indicates two different ways in which the vNW theorem may become non-applicable in the semiclassical limit.

6. Conclusion

In conclusion, we have found two anomalous monopoles, one line-shaped and one disk-shaped in a two-mode interacting boson system. These two anomalous monopoles represent two different ways in which the von Neumann–Wigner theorem fail in the semiclassical limit. In addition, the anomalous monopole of disk shape signals a breakdown of the Berry's semiclassical relation between quantum and classical two-forms.

We emphasize that even though these above results are obtained with a specific boson system. The results are expected to hold in a general interacting boson system. The reason is that our system is the simplest (or minimal) interacting boson system. This is analogous to the fact that most essential results with a two-level system, the simplest quantum system, hold up in a general quantum system. Our theoretical results can have experimental consequences. As mentioned at the beginning, the monopole in the Brillouin zone is found to be responsible for the anomalous Hall effect [8]. It is possible experimentally to set up to a Bose Hall system with a BEC in a rotating optical lattice, where the effect of the anomalous monopole is manifested. Another possibility is magnetic molecule [6,7], where the monopole may be found to play a role in the macroscopic quantum tunneling.

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