## Comment II on "Inconsistency in the Application of the Adiabatic Theorem"

Recently, Marzlin and Sanders (MS) [1] showed that there is some inconsistency in applying the quantum adiabatic theorem (QAT). Because of the importance of this theorem, this work has attracted a lot of attention [2,3]. We show here that MS made a mathematical mistake in arriving their conclusion.

In brief, MS considered an adiabatic system $H(t)$. For an eigenstate $|E_0(t_0)\rangle$ of this system at $t = t_0$, they defined a state $|\bar{\psi}\rangle \equiv U^\dagger(t, t_0)|E_0(t_0)\rangle$, where $U$ is the unitary evolution of the system $H(t)$. This state fulfills exactly the Schrödinger equation with a different Hamiltonian $\bar{H}(t) = -U^\dagger(t, t_0)H(t)U(t, t_0)$. MS went on to prove with Eq. (5) in Ref. [1] that the adiabatic approximation implies that $|\bar{\psi}\rangle \approx |\tilde{\psi}\rangle$, where $|\tilde{\psi}\rangle \equiv e^{i\int E_0}|E_0(t_0)\rangle$. Finally, they showed that the result $|\bar{\psi}\rangle \approx |\tilde{\psi}\rangle$ leads to an apparent contradictory, $1 \neq 1$. MS attributed it to the application of the QAT in Eq. (5), concluding that there is some inconsistency in applying the QAT.

However, MS made a mathematical mistake in arriving this conclusion. The contradictory is not caused by the QAT but by a lapse in MS's mathematical derivation. To get that contradictory, they need to prove $|\bar{\psi}\rangle \approx |\tilde{\psi}\rangle$ [claim Eq. (4) in Ref. [1]]. What was actually proved in (5) of Ref. [1] is that $|\tilde{\psi}\rangle$ satisfies approximately the Schrödinger equation with $\bar{H}$, a differential equation. However, one cannot prove claim (4) from Eq. (5) in Ref. [1] mathematically since, in general, an approximate differential equation can only lead to a valid solution in short time scales. When one integrates the approximate differential equation in Eq. (5), the error can accumulate over time and cause $|\tilde{\psi}\rangle$ deviates greatly from $|\bar{\psi}\rangle$. As a result, one cannot come to the contradictory Eq. (6) in Ref. [1]. We emphasize that whether one can obtain Eq. (4) from Eq. (5) has nothing to do with the QAT and it is purely mathematical.

As an example, we consider a two level model, $H(t) = -\frac{\omega_0}{2}\boldsymbol{\sigma}\cdot\boldsymbol{B}(t)$ with $\boldsymbol{B}(t) = \{\sin\theta\cos\omega t, \sin\theta\sin\omega t, \cos\theta\}$. This example was also studied in Ref. [2]. When $\omega \ll \omega_0$, this system is adiabatic. For the eigenstate $|E_1(t)\rangle$ corresponding to eigenvalue $\omega_0/2$, we find that

$$F \equiv |\langle\bar{\psi}|\tilde{\psi}\rangle| = \sqrt{1 - \sin^2\theta \sin^2\frac{\omega t}{2}}. \qquad (1)$$

It is clear that we have $F \approx 1$ for $t \ll 1$ and $F \neq 1$ for $t \gtrsim 1$. That is, $|\bar{\psi}\rangle \approx |\tilde{\psi}\rangle$ is correct only for short evolution times. More importantly, this example clearly demonstrates that the deviation shown in the above equation is not caused by the adiabatic approximation applied to $H(t)$. One can show with some algebra that for this $H(t)$ the deviation of its adiabatic solution from its exact solution is always bounded by a small value proportional to $\omega^2$ even for an infinite long evolution time. This implies that the long-time deviation in Eq. (1) is not caused by the adiabatic evolution.

One may argue that the mathematical lapse discussed above just illustrates that careless use of the adiabatic approximation can lead to erroneous results. However, if this were truly what MS meant, their statement would be trivial. Careless use of any approximation, not only the adiabatic approximation, can lead to errors.

MS used a counterexample, trying to demonstrate that there are more problems with the QAT. They showed that this example fulfills a widely-used adiabatic condition [Eq. (2) in Ref. [1]]; yet the adiabatic approximation does not hold. However, no one has proved that Eq. (2) in Ref. [1] is a sufficient adiabatic condition. Messiah in his well-known book [4] also discussed it after proving the QAT, stating clearly that it applies only "in most cases." In fact, MS's example contains a resonance term and cannot be an adiabatic system. MS also claimed that their counterexample fulfills more elaborate criteria such as the one in Ref. [5]. If so, there must be some subtle mistakes in these very rigorous mathematical proofs of the QAT. MS failed to point out any such mistake.

In summary, there is no "inconsistency" in applying the quantum adiabatic theorem; at least Marzlin and Sanders have not demonstrated this in Ref. [1] since there is a mathematical mistake in their reasoning and their counterexample is not an adiabatic system.

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[1] K.-P. Marzlin and B. C. Sanders, Phys. Rev. Lett. **93**, 160408 (2004).
[2] D. M. Tong, K. Singh, L. C. Kwek, and C. H. Oh, Phys. Rev. Lett. **95**, 110407 (2005).
[3] Zhaoyan Wu and Hui Yang, Phys. Rev. A**72**, 012114 (2005); R. Mackenzie, E. Marcotte, and H. Paquette, Phys. Rev. A **73**, 042104 (2006); Ming-Yong Ye, Xiang-Fa Zhou, Yong-Sheng Zhang, and Guang-Can Guo, quant-ph/0509083; Arun K. Pati and A. K. Rajagopal, quant-ph/0405129; M. S. Sarandy, L.-A. Wu, and D. A. Lidar, quant-ph/0405059.
[4] A. Messiah, *Quantum Mechanics* (Dover, New York, 1999), p. 754.
[5] T. Kato, J. Phys. Soc. Jpn. **5**, 435 (1950).

 