Superfluidity and stability of a Bose-Einstein condensate with periodically modulated interatomic interaction

Shao-Liang Zhang (张少良) and Zheng-Wei Zhou (周正威)
Key Laboratory of Quantum Information, University of Science and Technology of China, Hefei, Anhui 230026, China

Biao Wu (吴飙)*
International Center for Quantum Materials, Peking University, 100871 Beijing, China
(Received 20 December 2012; published 28 January 2013)

We study theoretically the superfluidity and stability of a Bose-Einstein condensate (BEC) whose interatomic scattering length is periodically modulated with optical Feshbach resonance. Our numerical study finds that the properties of this periodic BEC are strongly influenced by the modulation strength. When the modulation strength is small, only the Bloch waves close to the Brillouin zone edge suffer both Landau and dynamical instabilities. When the modulation strength is strong enough, all Bloch waves become dynamically unstable. In other words, the periodic BEC loses its superfluidity completely.

DOI: 10.1103/PhysRevA.87.013633 PACS number(s): 03.75.Kk, 05.30.Jp, 67.10.Ba

I. INTRODUCTION

Even though superfluidity is one of the most important macroscopic quantum phenomena, it could only be found in liquid helium before 1995. Since the realization of Bose-Einstein condensation in atomic gases in 1995, we have now in experiment another superfluid, the Bose-Einstein condensate (BEC) [1]. This new superfluid shares many interesting properties with superfluid helium, such as critical velocity [2] and quantized vortices [3]. At the same time, there are also some interesting properties unique to this gaseous superfluid; for example, there is no roton excitation in BECs. In particular, since a gas is easily compressible, one can modulate its density with an optical lattice to create a periodic superfluid. The properties of this periodic superfluid have been studied extensively both theoretically [4] and experimentally [5]. Dynamical instability, which is absent in a homogenous superfluid, was discovered and found to play a dominant role in destroying superfluidity in a periodic superfluid [4,5].

Now another type of periodic superfluid can be created: with optical Feshbach resonance (OFR) [6], the interatomic interaction (or scattering length) of a BEC can be modulated periodically in space with laser beams [7]. This was already demonstrated in experiment [8]. This BEC with periodically modulated interaction (PMI) is different from the usual BEC in an optical lattice, where the atoms feel an external periodic potential (PP). The most important difference between these two periodic BEC systems is that the BEC with PMI has no linear periodic counterpart. As a result, the widely used single-band approximation for a periodic system appears not applicable for a BEC with PMI. Furthermore, while the BEC in PP has a Mott-insulator phase [9,10], the BEC with PMI should not have the Mott phase. In this work we study the superfluidity and other related physical properties in a BEC with PMI.

We focus on the case where the laser beam for OFR is applied only along one direction. In this case, the BEC can be described by the following Gross-Pitaevskii equation:

$$i\hbar\frac{\partial\Psi}{\partial t} = -\frac{\hbar^2}{2m}\nabla^2\Psi + [V_1 + V_2\cos(2k_Z x)]|\Psi|^2\Psi, \quad (1)$$

where V_1 and V_2 are positive parameters that can be tuned experimentally by changing laser power and detuning [6–8]; m is the atomic mass and k_Z is the wave number of the laser beam. There have already been some theoretical attempts to find soliton solutions in such a system [11,12]. In this work we study the superfluidity of this periodic system. This is equivalent to examining the stability of a flow described by a Bloch wave [13]. The Bloch wave solutions of this BEC system for the lowest Bloch band are found numerically. Their Landau and dynamical stabilities are examined by computing the Bogoliubov excitations. We find that when the periodic modulation strength V_2 is small, only Bloch waves close to the Brillouin zone edge are unstable. When the modulation strength becomes large enough, all Bloch waves in the lowest band become dynamically unstable. This means that the periodic modulation of the scattering length can cause a BEC to lose its superfluidity completely. When this happens, the periodic Bose system is neither a superfluid nor a Mott insulator; the BEC may collapse into many solitons as suggested by an early study [14]. In contrast, for a BEC in PP, the Bloch state near the Brillouin zone center is always stable no matter how strong the periodic modulation is [4].

The paper is organized as follows. In Sec. II, we present the basic theoretical framework within which the BEC system is treated. In Sec. III, the lowest two Bloch bands for this periodic BEC system are presented and their physical meaning is discussed. In Sec. IV, we study the Landau instability and dynamical instability of the Bloch waves and discuss the superfluidity of this system. In Sec. V, the results in the previous section are discussed in the context of the experiment. The paper is summarized in Sec. VI.

II. THEORETICAL FRAMEWORK

We consider the one-dimensional case, where the scattering length is modulated only in one direction by optical Feshbach

*wubiao@pku.edu.cn

resonance and the lateral motion of the BEC can be either ignored or confined. In this case the Gross-Pitaevskii equation becomes one-dimensional:

$$i\frac{\partial}{\partial t}\psi = -\frac{1}{2}\frac{\partial^2}{\partial x^2}\psi + (c_1 + c_2\cos x)|\psi|^2\psi\,. \qquad (2)$$

Here the energy unit is $4\hbar^2 k_Z^2/m$, and the length unit is $1/2k_Z$. The wave function ψ is in units of $\sqrt{n_0}$, where n_0 is the averaged BEC density; $c_1 = \frac{mn_0V_1}{4\hbar^2k_Z^2}$, and $c_2 = \frac{mn_0V_2}{4\hbar^2k_Z^2}$. For convenience, we call c_1 the uniform strength and c_2 the modulation strength.

We are interested in the superfluidity and stability of a flow in this system. For a homogeneous BEC, the flow is described by a plane wave. For a periodic system, the flow is represented by a Bloch wave. The Bloch wave has the form $\psi(x) = e^{ikx}\varphi_k(x)$, where $\varphi_k(x)$ is of period 2π, it satisfies the time-independent Gross-Pitaevskii equation

$$\mu\psi = -\frac{1}{2}\frac{d^2}{dx^2}\psi + (c_1 + c_2\cos x)|\psi|^2\psi\,. \qquad (3)$$

We use the numerical method proposed in Ref. [15] to find these Bloch waves.

Once a Bloch wave solution is found, its stability is examined. We add a small perturbation to the Bloch wave $\varphi_k(x)$,

$$\delta\varphi_{k,q}(x) = u_k(x,q)e^{iqx} + v_k^*(x,q)e^{-iqx}, \qquad (4)$$

where q is in the range $[-1/2,1/2]$ and represents the mode of perturbation. The energy deviation caused by the perturbation is

$$\delta E_k = \int_{-\infty}^{\infty} dx(u_k^*,v_k^*)M_k(q)\begin{pmatrix}u_k\\ v_k\end{pmatrix}, \qquad (5)$$

where

$$M_k(q) = \begin{pmatrix}\mathcal{L}(k+q) & (c_1 + c_2\cos x)\varphi_k^2\\ (c_1 + c_2\cos x)\varphi_k^{*2} & \mathcal{L}(-k+q)\end{pmatrix} \qquad (6)$$

with

$$\mathcal{L}(k) = -\frac{1}{2}\left(\frac{\partial}{\partial x} + ik\right)^2 - \mu + 2(c_1 + c_2\cos x)|\varphi_k|^2\,. \qquad (7)$$

If the matrix $M_k(q)$ has negative eigenvalues, it means that there are some perturbations $\delta\varphi_{k,q}(x)$ that can lower the system energy. This energetic instability is related to the superfluidity of the system and we call it the Landau instability as this is the essence behind Landau's theory of superfluidity. The Landau instability is sometimes called a thermodynamical instability.

We also consider the dynamical evolution of the system after the perturbation $\delta\varphi_{k,q}$. For such a small perturbation, the dynamical Eq. (2) can be linearized and becomes

$$i\frac{\partial}{\partial t}\begin{pmatrix}u_k\\ v_k\end{pmatrix} = \sigma M_k(q)\begin{pmatrix}u_k\\ v_k\end{pmatrix}, \quad \sigma = \begin{pmatrix}I & 0\\ 0 & -I\end{pmatrix}. \qquad (8)$$

If matrix $\sigma M_k(q)$ has complex eigenvalues, the system becomes unstable and will damp during dynamical evolution. We call this instability dynamical instability.

Landau instability and dynamical instability can be discussed in a more coherent way within the framework of Bogoliubov excitations. The eigenvalues of matrix $\sigma M_k(q)$ can be divided into two groups, phonon modes and antiphonon modes; only phonon modes are physical and are usually called Bogoliubov excitations [15]. It can be proved that matrix $M_k(q)$ having a negative eigenvalue is equivalent to $\sigma M_k(q)$ having phonon modes of negative energy [16]. Therefore, the system has Landau instability when some of the phonon modes have negative energies and it has dynamical instability when some of the phonon modes are complex. As the phonon modes are related to the superfluidity of a Bose system [17], it is clear from this perspective that Landau instability and dynamical instability are clearly related to each other and are just two ways of destroying superfluidity. As we show, both instabilities are present in this periodic BEC.

III. NONLINEAR BLOCH BANDS

Because of the periodic modulation of interatomic interaction, this BEC has Bloch wave solutions. These Bloch waves can in general be found by numerically solving Eq. (3). However, near the edge of the Brillouin zone ($k \approx 1/2$), we can use a two-mode approximation and assume the Bloch state is of the form $\varphi_{\tilde{k}}(x) \approx ae^{i(\tilde{k}-\frac{1}{2})x} + be^{i(\tilde{k}+\frac{1}{2})x}$ ($|a|^2 + |b|^2 = 1$, $|\tilde{k}| \ll 1$ and $\tilde{k} = k - 1/2$). Plugging the trial wave function into Eq. (3), we can get a quartic equation [18]

$$w^4 + 2gw^3 + (g^2 - h^2 - 1)w^2 - 2gw - g^2 = 0, \qquad (9)$$

where $g = c_1/c_2$, $h = \tilde{k}/c_2$, $w = 1/(2ab)$, and the chemical potential is $\mu = 1.5c_1 + 0.5c_2(w + 1/w)$. It can be shown that near $\tilde{k} = 0$, Eq. (9) has only two real solutions when $g \leqslant 1$ ($c_1 \leqslant c_2$); it has four real solutions when $g > 1$ ($c_1 > c_2$). This means that the Bloch band of this nonlinear periodic system has a loop structure at the edge of the Brillouin zone when $g > 1$. This is confirmed by our numerical computation, as shown in Fig. 1. In Fig. 1(b), where $c_1 > c_2$, the chemical potential μ has a clear loop at the edge of the Brillouin zone.

The loop structure in the Bloch band in Fig. 1(b) is a manifestation of superfluidity in the system. We can consider the homogeneous case $c_2 = 0$. In this case, the BEC is a superfluid with critical velocity $v_c = \sqrt{c_1}$. Now we slowly turn on the periodic modulation by increasing c_2 to a small value. This small periodic modulation can be regarded as a

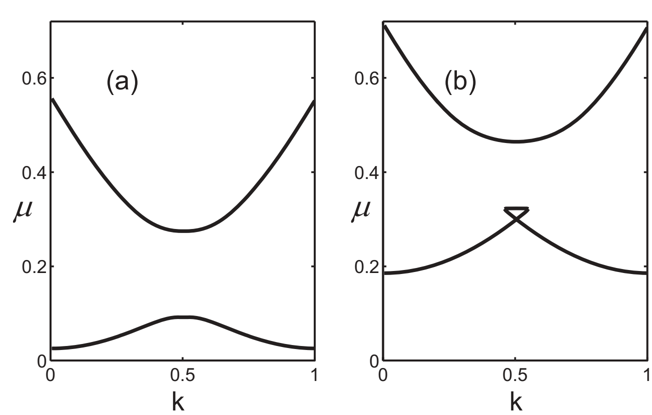

FIG. 1. The lowest two Bloch bands of a BEC with its interatomic interaction periodically modulated, $c_2 = 0.1$: (a) $c_1 = 0.05$, (b) $c_1 = 0.2$. μ is in units of $4\hbar^2k_Z^2/m$ and k is in units of $2k_Z$, where k_Z is the wave vector the laser beam.

perturbation when c_1 is large. The Bragg scattering caused by this periodic perturbation should not destroy a superflow moving with velocity $v = 1/2$, which is the velocity at the Brillouin zone edge when $c_2 \sim 0$. The nonzero slope of the Bloch band at the edge is an indication of this robustness of superfluidity. When the periodic modulation becomes very strong as in Fig. 1(a), the Bragg scattering can eventually disrupt the superflow as indicated by the zero slope at the zone edge in Fig. 1(a). A similar loop structure has been found in many different systems [19]. Note that the loop structure of the energy band has its interesting many-body counterpart, a net of narrow avoided crossings in energy levels [20].

IV. SUPERFLUIDITY AND INSTABILITY

For all the Bloch waves found in the lowest band, we have examined their superfluidity and stability by numerically computing their Bogoliubov excitations with $\sigma M_k(q)$. The results are shown in the stability phase diagram of Fig. 2. For a point (k,q) in the figure, there are three possibilities: (i) If it is in the white region, it means that the Bloch wave φ_k is a local energy minimum relative to the perturbation mode q. (ii) If it falls into the black area, the Bloch wave φ_k is dynamically unstable relative to the perturbation mode q. Any perturbation containing mode q will cause the system to evolve dynamically away from state φ_k with an exponential growth. (iii) If the point (k,q) lies in the gray region, the Bloch wave φ_k is not a local energy minimum but dynamically stable relative to the perturbation mode q. Note that we have only plotted the results for $k > 0$ in Fig. 2 as the system is symmetric with respect to time reversal and the results are the same for k and $-k$.

The stability phase diagram in Fig. 2(a) is very similar to the one for a BEC in PP [4]. However, as c_2 increases or c_1 decreases, the phase diagram begins to have new features. The most prominent is that the black area (dynamical instability) spreads into the region with $k < 1/4$ and eventually reaches $k = 0$. For a BEC in PP, the black area is restricted in the region with $k > 1/4$. It is also clear from the figure that the boundaries of the gray area and the black area coincide once the black area reaches $k = 0$. This means that once the Bloch state at $k = 0$ becomes unstable, it has both Landau instability and dynamical instability. In this case, the ground state of this system is no longer a Bloch wave. In contrast, the ground state of a BEC in PP is always a Bloch wave.

There is a qualitative way to understand why the Bloch state at $k = 0$ becomes dynamically unstable. When $c_1 < c_2$, the interaction between atoms becomes attractive for some parts of the system. As c_2 increases, a larger portion of the system becomes attractively interacting. Eventually, the overall interaction of the system, indicated by the averaged interaction over one period $V = \int_0^{2\pi}(c_1 + c_2\cos x)|\psi|^4 dx$, becomes negative. We expect that this is the underlying reason that the Bloch state at $k = 0$ becomes dynamically unstable. This is confirmed by our numerically computation. In Fig. 3, we have marked out the stability regions for the Bloch state at $k = 0$ in the parameter space of c_1 and c_2. We find numerically that the Bloch state at $k = 0$ is dynamically unstable in the black area in Fig. 3. The solid (red) line is the dividing line between averaged positive interaction and averaged negative interaction. From Fig. 3 we can see the solid (red) line is lower than the stability boundary for the Bloch state. It means that the Bloch state is always unstable when the averaged interaction is negative. However, this is not the only reason that the Bloch state becomes unstable as there is an area where the averaged interaction is positive while the Bloch state is unstable. This is yet to be fully understood.

We now approximate the periodic BEC with overall negative interaction with a homogeneous BEC with attractive interaction strength c. For a homogeneous BEC, the flow is described by plane waves e^{ikx}. In this case, the matrix $\sigma M_k(q)$ is a 2×2 matrix. The phonon excitations are easily obtained. For the state with $k = 0$, they are

$$\epsilon_{\rm ph} = \sqrt{q^2 c + q^4/4}. \tag{10}$$

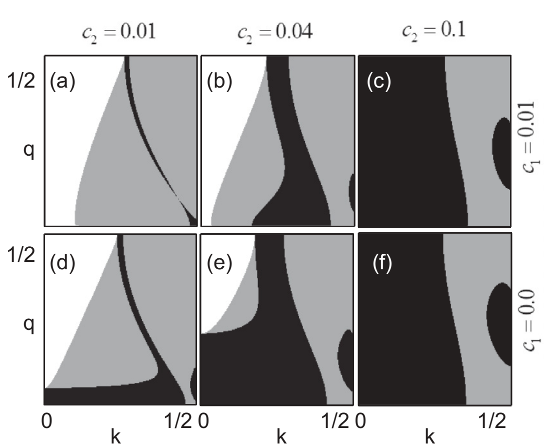

FIG. 2. Stability phase diagrams of BEC Bloch states for different values of the uniform strength c_1 and modulation strength c_2. k is the wave number of the Bloch state and q is the wave number of the perturbation mode. In the white area, the Bloch state is a local energy minimum; in the shaded (light or dark) area, the Bloch state has negative excitation energy and Landau instability; in the dark shaded area, the Bloch state is dynamically unstable. k and q are in units of $2k_Z$, where k_Z is the wave vector of the laser beam.

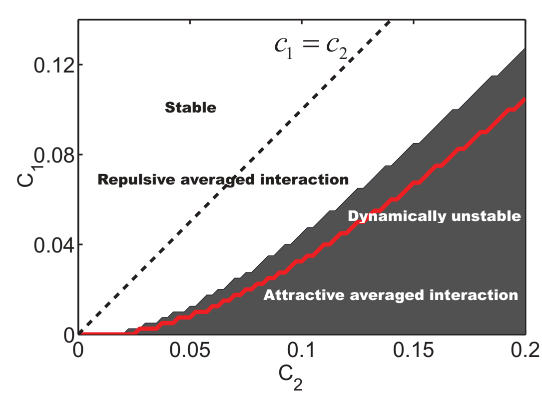

FIG. 3. (Color online) The stability phases of the Bloch state at $k = 0$ in the space of the interaction parameters c_1 and c_2. The Bloch wave state is stable in the white area; it is dynamically unstable in the black areas. On the solid (red) line, the averaged interatomic interaction is zero; below it, the averaged interaction becomes negative; and above it, the averaged interaction is positive.

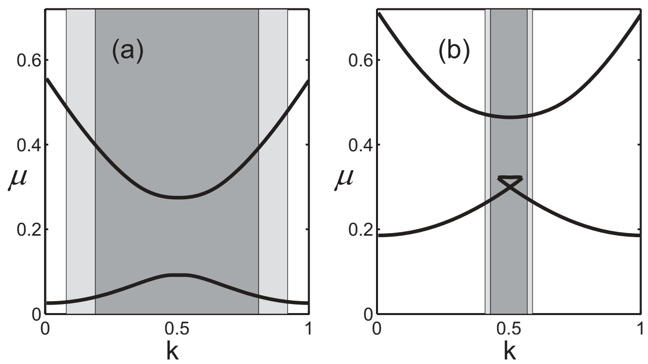

FIG. 4. Stabilities of the Bloch waves in the lowest Bloch bands shown in Fig. 1. The Bloch states in the black area are dynamically unstable, and the Bloch states in the shaded area have Landau instability, $c_2 = 0.1$: (a) $c_1 = 0.05$, (b) $c_1 = 0.2$. μ is in units of $4\hbar^2 k_Z^2/m$ and k is in units of $2k_Z$, where k_Z is the wave vector of the laser beam.

Since c is negative, the phonon excitation is imaginary for small q. This is in fact the feature seen in Figs. 2(c)–2(f): the Bloch state at $k = 0$ is dynamically unstable against the perturbation of mode $q = 0$. This approximation result further confirms that the averaged interaction plays a dominant role in this system of a BEC with PMI and is the key factor in stability and superfluidity of the BEC system.

V. EXPERIMENTAL PERSPECTIVE

The superfluidity and dynamical instability has been explored experimentally for a BEC in PP [5]. A similar experimental scheme can be used to study superfluidity and dynamical instability of this periodic BEC system. There exists no fundamental technical barrier.

In a real experiment, there is no way to control the perturbation mode. The controlled perturbation and the uncontrollable noises in the experiment should contain all the possible modes of q. Therefore, for an experimentalist, a Bloch state φ_k is unstable if it is unstable against any of the perturbation modes. For the Bloch waves in the lowest band in Fig. 1, we have marked out their stabilities with gray and black shadings in Fig. 4. It is clear from the figure that the Bloch states near the Brillouin zone center are more stable and more Bloch states become stable as the uniform strength c_1 increases.

There exist two critical values of k in Fig. 4, k_L and k_D. A BEC Bloch wave φ_k with k in the range $[k_L,1/2]$ has Landau instability; it has dynamical instability if its k is in the range $[k_D,1/2]$. Both critical values k_L and k_D vary with c_1 and c_2, and this dependence is shown in Fig. 5. As seen in Fig. 5(a), k_L increases with the increase of c_1 and the decrease of c_2. In Fig. 5(b), k_D is discontinuous at the edge of $c_2 = 0$. This is due to that at $c_2 = 0$, the system loses the periodicity and it has no dynamical instability. On the two (red) lines Fig. 5,

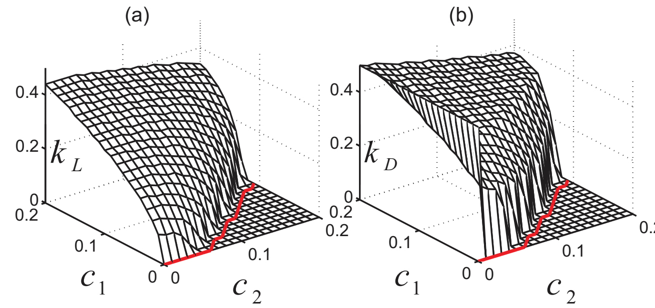

FIG. 5. (Color online) The critical values of k_L and k_D as a function of c_1 and c_2. k_L is the critical Bloch wave number for the Landau instability and k_D is the critical Bloch wave number for the dynamical instability. The (red) lines are where k_L and k_D turn to zero. The two (red) lines in (a) and (b) are identical. k_L and k_D are in units of $2k_Z$, where k_Z is the wave vector of the laser beam.

both k_L and k_D turn zero. The two (red) lines are the same. This reflects a fact that we already mentioned: when the Bloch wave at $k = 0$ has Landau instability, it is also dynamically unstable.

VI. SUMMARY

We have studied a BEC with its scattering length periodically modulated in one direction. In this periodic BEC system, the flows are represented by Bloch waves and the energy has band structure. When the uniform repulsive interaction is larger than the periodic modulation strength, the band structure has a loop at the edge of the first Brillouin zone. We have also studied the stabilities of these Bloch waves. When the modulation strength is weak, this system is similar to a BEC in PP. When the modulation strength is strong enough, even the Bloch state at the Brillouin zone center becomes unstable. This means that the BEC loses it superfluidity completely.

ACKNOWLEDGMENTS

This work was funded by the National Natural Science Foundation of China (Grants No. 11174270 and No. 60921091), the National Basic Research Program of China (Grant No. 2011CB921204), the China Postdoctoral Science Foundation No. 2011M501384, the Fundamental Research Funds for the Central Universities (Grant No. WK2470000006), and the Research Fund for the Doctoral Program of Higher Education of China (Grant No. 20103402110024). B.W. is supported by the NBRP of China (Grants No. 2012CB921300 and No. 2013CB921900) and the NSF of China (Grants No. 10825417, No. 11274024, and No. 11128407), and the RFDP of China (Grant No. 20110001110091). Z.-W.Z. gratefully acknowledges the support of the K. C. Wong Education Foundation, Hong Kong.

[1] M. H. Anderson, J. R. Ensher, M. R. Matthews, C. E. Wieman, and E. A. Cornell, Science, **269**, 198 (1995); C. C. Bradley, C. A. Sackett, J. J. Tollett, and R. G. Hulet, Phys. Rev. Lett. **75**, 1687 (1995); K. B. Davis, M.-O. Mewes, M. R. Andrews, N. J. van Druten, D. S. Durfee, D. M. Kurn, and W. Ketterle, *ibid.* **75**, 3969 (1995).

[2] C. Raman, M. Köhl, R. Onofrio, D. S. Durfee, C. E. Kuklewicz, Z. Hadzibabic, and W. Ketterle, Phys. Rev. Lett. **83**, 2502 (1999).
[3] K. W. Madison, F. Chevy, W. Wohlleben, and J. Dalibard, Phys. Rev. Lett. **84**, 806 (2000); J. R. Abo-Shaeer, C. Raman, J. M. Vogels, and W. Ketterle, Science **292**, 476 (2001); D. L. Feder and C. W. Clark, Phys. Rev. Lett. **87**, 190401 (2001); E. Hodby, G. Hechenblaikner, S. A. Hopkins, O. M. Maragò, and C. J. Foot, *ibid.* **88**, 010405 (2001).
[4] K. Berg-Sørensen and K. Mølmer, Phys. Rev. A **58**, 1480 (1998); D. I. Choi and Q. Niu, Phys. Rev. Lett. **82**, 2022 (1999); B. Wu and Q. Niu, Phys. Rev. A **64**, 061603(R) (2001); E. J. Mueller, *ibid.* **66**, 063603 (2002); A. Polkovnikov, E. Altman, E. Demler, B. Halperin, and M. D. Lukin, *ibid.* **71**, 063613 (2005); V. I. Yukalov, Laser Phys. **19**, 1 (2009).
[5] O. Morsch, J. H. Müller, M. Cristiani, D. Ciampini, and E. Arimondo, Phys. Rev. Lett. **87**, 140402 (2001); L. Fallani, L. De Sarlo, J. E. Lye, M. Modugno, R. Saers, C. Fort, and M. Inguscio, *ibid.* **93**, 140406 (2004); O. Morsch and M. Oberthaler, Rev. Mod. Phys. **78**, 179 (2006).
[6] M. Theis, G. Thalhammer, K. Winkler, M. Hellwig, G. Ruff, R. Grimm, and J. H. Denschlag, Phys. Rev. Lett. **93**, 123001 (2004).
[7] R. Qi and H. Zhai, Phys. Rev. Lett. **106**, 163201 (2011).
[8] R. Yamazaki, S. Taie, S. Sugawa, and Y. Takahashi, Phys. Rev. Lett. **105**, 050405 (2010).
[9] D. Jaksch, C. Bruder, J. I. Cirac, C. W. Gardiner, and P. Zoller, Phys. Rev. Lett. **81**, 3108 (1998).
[10] M. Greiner, O. Mandel, T. Esslinger, T. W. Hänsch, and I. Bloch, Nature (London) **415**, 39 (2002).
[11] Hidetsugu Sakaguchi and Boris A. Malomed, Phys. Rev. E **72**, 046610 (2005); N. V. Hung, P. Ziń, M. Trippenbach, and B. A. Malomed, *ibid.* **82**, 046602 (2010).
[12] F. Abdullaev, A. Abdumalikov, and R. Galimzyanov, Phys. Lett. A **367**, 149 (2007); F. Kh. Abdullaev, A. Gammal, H. L. F. da Luz, and L. Tomio, Phys. Rev. A **76**, 043611 (2007)
[13] B. Wu and J. Shi, arXiv:cond-mat/0607098; Q. Zhu, C. Zhang, and B. Wu, Europhys. Lett. **100**, 50003 (2012).
[14] Z. W. Zhou, S. L. Zhang, X. F. Zhou, G. C. Guo, X. X. Zhou, and H. Pu, Phys. Rev. A **83**, 043626 (2011).
[15] B. Wu and Q. Niu, New J. Phys. **5**, 104 (2003).
[16] Z. Chen and B. Wu, Phys. Rev. A **81**, 043611 (2010).
[17] E. M. Lifshitz and L. P. Pitaevskii, *Statistical Physics (Part 2)* (Pergamon Press Inc., New York).
[18] B. Wu and Q. Niu, Phys. Rev. A **61**, 023402 (2000).
[19] B. Wu, R. B. Diener, and Q. Niu, Phys. Rev. A **65**, 025601 (2002); D. Diakonov, L. M. Jensen, C. J. Pethick, and H. Smith, *ibid.* **66**, 013604 (2002); M. Machholm, C. J. Pethick, and H. Smith, *ibid.* **67**, 053613 (2003); B. T. Seaman, L. D. Carr, and M. J. Holland, *ibid.* **71**, 033622 (2005); G. Watanabe, S. Yoon, and F. Dalfovo, Phys. Rev. Lett. **107**, 270404 (2011); B. Prasanna Venkatesh, J. Larson, and D. H. J. O'Dell, Phys. Rev. A **83**, 063606 (2011); C.-S. Chien, S.-L. Chang, and B. Wu, Comput. Phys. Commun. **181**, 1727 (2010); Z. Chen and B. Wu, Phys. Rev. Lett. **107**, 065301 (2011); H.-Y. Hui, R. Barnett, J. V. Porto, and S. Das Sarma, Phys. Rev. A **86**, 063636 (2012).
[20] Z. P. Karkuszewski, K. Sacha, and A. Smerzi, Eur. Phys. J. D **21**, 251 (2002); Biao Wu and Jie Liu, Phys. Rev. Lett. **96**, 020405 (2006); E. M. Graefe and H. J. Korsch, Phys. Rev. A **76**, 032116 (2007).