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Vlasov equation of plasma in magnetic field

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Abstract. The linearized Vlasov equation for a plasma system in a constant external magnetic field and the corresponding linear Vlasov operator are studied. The solution of the Vlasov equation is found by the resolvent method. The spectrum and eigenfunctions of the Vlasov operator are also found. The spectrum of this operator consists of two parts: one is continuous and real; the other is discrete and complex. Interestingly, the real eigenvalues are uncountably infinitely degenerate, which causes difficulty in solving this initial value problem by using the conventional eigenfunction expansion method. It also breaks the natural relation between the eigenfunctions and the resolvent solution in which the eigenfunctions can normally be considered as the coefficients of $e^{-i\omega t}$ in the Laplace (or resolvent) solution.

1. Introduction

In plasma physics, many interesting phenomena such as plasma oscillations, instabilities and Landau dampings can be studied and understood through a very simple plasma model. In this model, ions are assumed to be motionless and to form a positively charged background and the collisions between electrons are neglected. In 1945, Vlasov proposed a nonlinear equation, now known as the Vlasov equation, to describe this model [1]. Since then, much effort has been devoted to the study of this equation. When there is no external field, one usual way of treating this problem is to first linearize the equation by assuming the system is very close to the equilibrium state, then reduce it to a one-dimensional equation. Many interesting phenomena can be understood through this simple one-dimensional linearized Vlasov equation [2–4].

In this paper, I shall study this system in a constant external magnetic field. After the Vlasov equation is linearized, two methods, the eigenfunction expansion method and the resolvent method, are tried to solve the equation. The resolvent method is proved to be successful. Surprisingly, the eigenfunction expansion method is not successful, even after all the eigenfunctions of the corresponding linear operator, the Vlasov operator, have been worked out. The difficulty is caused by the fact that the real eigenvalues are uncountably infinitely degenerate.

The explicit relation between the Laplace transform (or the resolvent) approach and the eigenfunction expansion approach to the one-dimensional Vlasov equation with no external field was studied by Case [5] and Arthur *et al* [6]. In particular, the latter demonstrated how to construct the van Kampen–Case modes [5, 7] through the resolvent solution. Their results show that the van Kampen–Case modes are, loosely speaking, simply the coefficients of $e^{i\omega t}$ in the resolvent (or Laplace) solution. This is no longer true when we do not reduce the Vlasov equation to a one-dimensional problem or we simply cannot because of the existence of external fields. The eigenmodes (or eigenfunctions) corresponding to the real eigenvalues

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presented in section 3 cannot be constructed with the resolvent solution in section 5, which is caused by the infinite degeneracy mentioned above.

The plan of this paper is as follows. In section 2, the derivation of the linearized Vlasov equations and their related linear Vlasov operators are sketched, primarily to introduce notation. The eigenfunctions are presented and discussed in section 3 without detailed calculations. Next, in section 4, the related adjoint problem is discussed briefly and the orthogonal relation between the original eigenfunctions and the adjoint eigenfunctions is proved. In section 5, the exact solution is given by using the resolvent method.

2. The Vlasov equations and Vlasov operators

This simple plasma model is described by the Vlasov equation

$$\frac{\partial}{\partial t}\rho(\vec{x},\vec{v},t) = -\vec{v}\cdot\frac{\partial}{\partial\vec{x}}\rho(\vec{x},\vec{v},t) - \frac{\vec{F}}{m_e}\cdot\frac{\partial}{\partial\vec{v}}\rho(\vec{x},\vec{v},t) \tag{1}$$

where $\rho(\vec{x},\vec{v},t)$ is the single distribution function of electrons and $\vec{F}$ is the force felt by a single electron. In this paper, the system is in a constant external magnetic field $\vec{B}_0$, so

$$\vec{F} = -e\vec{v}\times\vec{B}_0 - e\vec{E} \tag{2}$$

where

$$\vec{E}(\vec{x},t) = e\frac{\partial}{\partial\vec{x}}\int \mathrm{d}\vec{x}'\,\mathrm{d}\vec{v}'\,\frac{\rho(\vec{x}',\vec{v}',t)}{|\vec{x}-\vec{x}'|}$$

is the electric field at position $\vec{x}$ and time t generated by the electron system itself. If the system is just slightly away from an equilibrium state $f_0(\vec{v})$

$$\rho(\vec{x},\vec{v},t) = n_0 f_0(\vec{v}) + f(\vec{x},\vec{v},t) \tag{3}$$

where n_0 is the average density and $f \ll n_0 f_0$, then it can be described very well by the linearized Vlasov equation, which is

$$\frac{\partial f}{\partial t} = -\vec{v}\cdot\frac{\partial f}{\partial\vec{x}} + \frac{e}{m_e}(\vec{v}\times\vec{B}_0)\cdot\frac{\partial f}{\partial\vec{v}} + \frac{e^2 n_0}{m_e}\left[\frac{\partial}{\partial\vec{x}}\int \mathrm{d}\vec{x}'\,\mathrm{d}\vec{v}'\,\frac{f}{|\vec{x}-\vec{x}'|}\right]\cdot\frac{\partial f_0}{\partial\vec{v}}. \tag{4}$$

Taking the Fourier transform

$$f(\vec{x},\vec{v},t) = \sum_{\vec{k}\neq 0} f_{\vec{k}}(\vec{v},t)\exp(\mathrm{i}\vec{k}\cdot\vec{x}) \tag{5}$$

for the $\vec{k}$-component function $f_{\vec{k}}(\vec{v},t)$ we have

$$\mathrm{i}\frac{\partial}{\partial t}f_{\vec{k}}(\vec{v},t) = \vec{k}\cdot\vec{v}\,f_{\vec{k}}(\vec{v},t) + \mathrm{i}\frac{e}{m_e}(\vec{v}\times\vec{B}_0)\cdot\frac{\partial}{\partial\vec{v}}f_{\vec{k}}(\vec{v},t) - \frac{\omega_p^2}{k^2}\vec{k}\cdot\frac{\partial}{\partial\vec{v}}f_0(\vec{v})\int \mathrm{d}\vec{v}'\,f_{\vec{k}}(\vec{v}',t) \tag{6}$$

where $\omega_p = (4\pi e^2 n_0/m_e)^{1/2}$ is the plasma frequency. To simplify the notation, let us drop the index $\vec{k}$ and let

$$\eta(\vec{v}) = \frac{\omega_p^2}{k^2}\vec{k}\cdot\frac{\partial}{\partial\vec{v}}f_0(\vec{v}) \tag{7}$$

then equation (6) becomes

$$\mathrm{i}\frac{\partial}{\partial t}f(\vec{v},t) = \vec{k}\cdot\vec{v}\,f(\vec{v},t) + \mathrm{i}\frac{e}{m_e}(\vec{v}\times\vec{B}_0)\cdot\frac{\partial}{\partial\vec{v}}f(\vec{v},t) - \eta(\vec{v})\int \mathrm{d}\vec{v}'\,f(\vec{v}',t). \tag{8}$$

This is the equation I shall study mostly in this paper. Consequently, the corresponding linear Vlasov operator $\mathcal{K}$ is defined as

$$\mathcal{K}f(\vec{v}) = \vec{k}\cdot\vec{v}f(\vec{v}) + \mathrm{i}\frac{e}{m_e}(\vec{v}\times\vec{B}_0)\cdot\frac{\partial}{\partial\vec{v}}f(\vec{v}) - \eta(\vec{v})\int \mathrm{d}\vec{v}'\, f(\vec{v}'). \tag{9}$$

This is an integro-differential, unbounded and non-self-adjoint linear operator. In terms of $\mathcal{K}$, equation (8) can be put into a compact form

$$\mathrm{i}\frac{\partial}{\partial t}f(\vec{v},t) = \mathcal{K}f(\vec{v},t). \tag{10}$$

The equation for the case where $\vec{B}_0 = 0$ is

$$\mathrm{i}\frac{\partial}{\partial t}f(\vec{v},t) = \mathcal{K}_0 f(\vec{v},t) \tag{11}$$

with

$$\mathcal{K}_0 f(\vec{v}) = \vec{k}\cdot\vec{v}f(\vec{v}) - \eta(\vec{v})\int \mathrm{d}\vec{v}'\, f(\vec{v}'). \tag{12}$$

Integrating over the two components of $\vec{v}$ perpendicular to $\vec{k}$, we can reduce (11) to a one-dimensional problem

$$\mathrm{i}\frac{\partial}{\partial t}\bar{f}(v,t) = \bar{\mathcal{K}}_0\bar{f}(v,t) \tag{13}$$

with

$$\bar{\mathcal{K}}_0\bar{f}(v) = kv\bar{f}(v) - \bar{\eta}(v)\int_{-\infty}^{\infty}\mathrm{d}v'\,\bar{f}(v'). \tag{14}$$

Here v is the component of $\vec{v}$ along the direction of $\vec{k}$ and k is the magnitude of $\vec{k}$. The bar over functions indicates that they have been integrated over the two components of $\vec{v}$ perpendicular to $\vec{k}$. This convention is followed throughout this paper. Equation (13) has been studied thoroughly by van Kampen, Case and many others [2, 5–8].

Formally, equations (10), (11) and (13) are just the same as the Schrödinger equation. However, the Hamiltonian operator appearing in the Schrödinger equation is self-adjoint and bounded from the below, while $\bar{\mathcal{K}}_0$, $\mathcal{K}_0$ and $\mathcal{K}$ are unbounded and not self-adjoint. These are all typical equations arising in linear evolution systems. There are many standard methods for studying this class of linear equations. For example, equation (13) can be solved by the Laplace transform [2,5], the resolvent method [6] and the eigenfunction expansion method [5,7]. I shall try to use the eigenfunction expansion method and the resolvent method to solve equations (10) and (11).

3. Eigenfunctions and eigenvalues

In this section, I shall present the eigenfunctions and spectra of operators $\bar{\mathcal{K}}_0$, $\mathcal{K}_0$ and $\mathcal{K}$. These three operators are similar in many respects as they are supposed to be. Their spectra are the same: continuous real eigenvalues, discrete real eigenvalues and discrete complex eigenvalues. Since the existence of discrete real eigenvalues depends on the choice of the equilibrium function $f_0(\vec{v})$, I shall only consider, for simplicity, the case where the discrete real eigenvalues do not exist. Also, their eigenfunctions are similar: most of them are singular containing δ functions. However, there is one major difference: the real eigenvalues of $\mathcal{K}_0$ and $\mathcal{K}$ are infinitely degenerate, while the eigenvalues of $\bar{\mathcal{K}}_0$ are not degenerate. This infinite degeneracy causes the difficulty in expanding the functions in terms of these eigenfunctions and relating these eigenfunctions with the resolvent solutions.

3.1. The operator $\bar{\mathcal{K}}_0$

The eigenfunctions of $\bar{\mathcal{K}}_0$ were first constructed by van Kampen [7] and then completed by Case [5], and are now widely known as van Kampen–Case modes. Their results can be summarized as follows.

The eigenequation for the eigenvalue z is

$$\bar{\mathcal{K}}_0\bar{g}_z(v) = kv\bar{g}_z(v) - \bar{\eta}(v)\int_{-\infty}^{\infty} \mathrm{d}v'\, \bar{g}_z(v') = z\bar{g}_z(v). \tag{15}$$

It is linear, so the functions can be normalized as

$$\int_{-\infty}^{\infty} \mathrm{d}v\, \bar{g}_z(v) = 1. \tag{16}$$

When the eigenvalue is real, say, it is ν, the eigenfunction is

$$\bar{g}_\nu(v) = \mathcal{P}\frac{\bar{\eta}(v)}{kv-\nu} + \bar{\lambda}(v)\delta(kv-\nu). \tag{17}$$

Here $\mathcal{P}$ means the principal value integral. The normalization condition (16) requires

$$\frac{1}{k}\bar{\lambda}(\frac{\nu}{k}) = 1 - \int \mathrm{d}v\, \mathcal{P}\frac{\bar{\eta}(v)}{kv-\nu}. \tag{18}$$

The discrete complex eigenvalues ν_j $(j = 1, 2, \ldots, m_0)$ are determined by

$$\epsilon_0(z) = 1 - \int_{-\infty}^{\infty} \mathrm{d}v\, \frac{\bar{\eta}(v)}{kv-z} = 1 - \int \mathrm{d}\vec{v}\, \frac{\eta(\vec{v})}{\vec{k}\cdot\vec{v}-z} = 0. \tag{19}$$

The corresponding eigenfunction is

$$g_j(v) = g_{\nu_j}(v) = \frac{\bar{\eta}(v)}{kv-\nu_j} \tag{20}$$

which satisfies the normalization condition (16). the eigenfunctions (17) and (20) are the famous van Kampen–Case modes. It is clear that all the eigenvalues of $\bar{\mathcal{K}}_0$ are not degenerate when the van Kampen–Case modes are normalized according to (16).

3.2. The operator $\mathcal{K}_0$

The eigenequation here is

$$\mathcal{K}_0 g_z(\vec{v}) = \vec{k}\cdot\vec{v}\, g_z(\vec{v}) - \eta(\vec{v})\int \mathrm{d}\vec{v}'\, g_z(\vec{v}') = z g_z(\vec{v}). \tag{21}$$

As this operator is linear, we set the normalization condition

$$\int \mathrm{d}\vec{v}\, g_z(\vec{v}) = 1. \tag{22}$$

The eigenfunction corresponding to a real eigenvalue ν is

$$g_\nu(\vec{v}) = \mathcal{P}\frac{\eta(\vec{v})}{\vec{k}\cdot\vec{v}-\nu} + \lambda(\vec{v})\delta(\vec{k}\cdot\vec{v}-\nu). \tag{23}$$

The normalization condition (22) requires that $\lambda(\vec{v})$ satisfy

$$\int \mathrm{d}\vec{v}\, \lambda(\vec{v})\delta(\vec{k}\cdot\vec{v}-\nu) = 1 - \int \mathrm{d}\vec{v}\, \mathcal{P}\frac{\eta(\vec{v})}{\vec{k}\cdot\vec{v}-\nu}. \tag{24}$$

It is very clear here that there are infinitely many choices for $\lambda(\vec{v})$ to satisfy the above condition. For example, all the functions $\lambda(\vec{v}) = \alpha(v_1, v_2)\bar{\lambda}(v)$ satisfy (24) as long as

$$\int_{-\infty}^{\infty} \mathrm{d}v_1 \int_{-\infty}^{\infty} \mathrm{d}v_2\, \alpha(v_1, v_2) = 1 \tag{25}$$

where v_1 and v_2 are the two components of $\vec{v}$ perpendicular to $\vec{k}$. Obviously, the choices of $\alpha(v_1, v_2)$ are uncountably infinite, which means there are infinitely many corresponding eigenfunctions for any real eigenvalue ν. In other words, the degeneracy of every real eigenvalue is uncountably infinite.

The complex eigenvalues of $\mathcal{K}_0$ are the same as $\bar{\mathcal{K}}_0$, ν_j $(j = 1, 2, \ldots, m_0)$, the zeros of $\epsilon_0(z)$, whose corresponding eigenfunctions are

$$g_j(\vec{v}) = g_{\nu_j}(\vec{v}) = \frac{\eta(\vec{v})}{\vec{k}\cdot\vec{v} - \nu_j}. \tag{26}$$

3.3. The operator $\mathcal{K}$

In this case I shall present the results directly without going into the detailed calculations. The method of computing eigenfunctions is very similar to that given in the appendix. First, let us set up a coordinate system and introduce some notation:

$$\vec{B}_0 = B_0\hat{z} \qquad \vec{k} = k_\perp\hat{x} + k_\parallel\hat{z} \tag{27}$$

$$\vec{v} = v_\perp\cos\theta\hat{x} + v_\perp\sin\theta\hat{y} + v_\parallel\hat{z}. \tag{28}$$

Thus the operator $\mathcal{K}$ becomes

$$\mathcal{K}f(\vec{v}) = (k_\perp v_\perp\cos\theta + k_\parallel v_\parallel)f(\vec{v}) - \mathrm{i}\omega_0\frac{\partial}{\partial\theta}f(\vec{v}) - \eta(\vec{v})\int \mathrm{d}\vec{v}'\, f(\vec{v}') \tag{29}$$

where $\omega_0 = eB_0/m_e$ is the cyclotron frequency. The same normalization condition

$$\int \mathrm{d}\vec{v}\, G_z(\vec{v}) = 1 \tag{30}$$

is set for the eigenequation

$$\mathcal{K}G_z(\vec{v}) = zG_z(\vec{v}). \tag{31}$$

As the complex eigenvalues of $\mathcal{K}_0$ are determined by $\epsilon_0(z) = 0$, the complex eigenvalues z_j $(j = 1, 2, \ldots, m)$ of $\mathcal{K}$ are determined by $\epsilon(z) = 0$, where

$$\epsilon(z) = 1 + \pi\sum_n \int_0^\infty v_\perp\,\mathrm{d}v_\perp \int_{-\infty}^\infty \mathrm{d}v_\parallel\, \frac{J_nJ_{n-1}\eta_\perp + J_nJ_{n+1}\eta_\perp + 2J_n^2\eta_\parallel}{2(z - m\omega_0 - k_\parallel v_\parallel)}. \tag{32}$$

Here and after, all summations are assumed to be over all integers. The J_n are the Bessel functions $J_n(k_\perp v_\perp/\omega_0)$. Two functions, $\eta_\perp$ and $\eta_\parallel$, are defined as

$$\eta_\perp = \frac{\omega_p^2}{k^2}k_\perp\frac{\partial}{\partial v_\perp}f_0(\vec{v}) \qquad \eta_\parallel = \frac{\omega_p^2}{k^2}k_\parallel\frac{\partial}{\partial v_\parallel}f_0(\vec{v}). \tag{33}$$

Note the equilibrium state function $f_0(\vec{v})$ has been assumed to be a function of $v_\perp$ and $v_\parallel$ only, and m, the number of the discrete complex eigenvalues, can be infinite. The eigenfunction corresponding to z_j is

$$G_j(\vec{v}) = \exp\left(\mathrm{i}\frac{k_\perp v_\perp\sin\theta}{\omega_0}\right)\sum_n \mathrm{e}^{-\mathrm{i}n\theta}\frac{(J_{n+1} + J_{n-1})\eta_\perp/2 + J_n\eta_\parallel}{z_j - n\omega_0 - k_\parallel v_\parallel}. \tag{34}$$

For a real eigenvalue μ, we have

$$G_\mu(\vec{v}) = \exp\left(\mathrm{i}\frac{k_\perp v_\perp\sin\theta}{\omega_0}\right)\sum_n \mathrm{e}^{-\mathrm{i}n\theta}\mathcal{P}\frac{(J_{n+1} + J_{n-1})\eta_\perp/2 + J_n\eta_\parallel}{\mu - n\omega_0 - k_\parallel v_\parallel}$$
$$+ \sum_n a_n(v_\parallel, v_\perp)\exp -\mathrm{i}\left(\frac{k_\perp v_\perp\sin\theta}{\omega_0} - n\theta\right)\delta(\mu - n\omega_0 - k_\parallel v_\parallel) \tag{35}$$

where the a_n are arbitrary functions as long as they satisfy the only constraint, the normalization condition (30). This means that there are infinitely many eigenfunctions $G_\mu(\vec{v})$ corresponding to each real eigenvalue μ. The infinite degeneracy arises again. It is easy to check that all these expressions reduce to those for $\vec{B}_0 = 0$.

We see that there are many similarities among $\bar{\mathcal{K}}_0$, $\mathcal{K}_0$ and $\mathcal{K}$. However, there is one important difference: the real eigenvalues of $\bar{\mathcal{K}}_0$ are not degenerate, while the real eigenvalues of both $\mathcal{K}_0$ and $\mathcal{K}$ are uncountably infinitely degenerate. As shown by Case [5], the solution of equation (13) can be expanded in terms of the van Kampen–Case modes. In other words, we can use the eigenfunction expansion method to solve equation (13) just as we often do to solve the Schrödinger equation. However, this method is not suitable for solving equations (10) and (11), even though we know the eigenfunctions of $\mathcal{K}_0$ and $\mathcal{K}$. Due to the infinite degeneracy, there is no obvious way that functions can be expanded in terms of these eigenfunctions. It is a common belief that for an initial value problem of a linear evolution system, to finding the solution is equivalent to find the eigenfunctions and spectrum of the linear operator in this system. Seemingly, this is not the case for $\mathcal{K}_0$ and $\mathcal{K}$.

4. Adjoint equations and operators

Operator $\mathcal{K}$ is not self-adjoint, so it is interesting to know the functions orthogonal to its eigenfunctions. For this purpose, we consider the adjoint equation to (10)

$$\mathrm{i}\frac{\partial}{\partial t} f_(\vec{v}, t) = \tilde{\mathcal{K}} f(\vec{v}) \tag{36}$$

where the linear operator $\tilde{\mathcal{K}}$ is defined as

$$\tilde{\mathcal{K}} f(\vec{v}) = \vec{k}\cdot\vec{v} f(\vec{v}) + \mathrm{i}\frac{e}{m_e}(\vec{v}\times\vec{B}_0)\cdot\frac{\partial}{\partial\vec{v}} f(\vec{v}) - \int \mathrm{d}\vec{v}'\,\eta(\vec{v}') f(\vec{v}'). \tag{37}$$

The discrete complex eigenvalues of $\tilde{\mathcal{K}}$ are also determined by equation (32). Its eigenfunction corresponding to z_j is

$$\tilde{G}_j(\vec{v}) = \exp\left(\mathrm{i}\frac{k_\perp v_\perp \sin\theta}{\omega_0}\right)\sum_n \mathrm{e}^{-\mathrm{i}n\theta}\frac{J_n}{z_j - n\omega_0 - k_\parallel v_\parallel} \tag{38}$$

while its eigenfunction corresponding to a real eigenvalue μ is

$$\tilde{G}_\mu(\vec{v}) = \exp\left(\mathrm{i}\frac{k_\perp v_\perp \sin\theta}{\omega_0}\right)\sum_n \mathrm{e}^{-\mathrm{i}n\theta}\mathcal{P}\frac{J_n}{\mu - n\omega_0 - k_\parallel v_\parallel} + \sum_n \tilde{a}_n(v_\parallel, v_\perp)\exp -\mathrm{i}\left(\frac{k_\perp v_\perp \sin\theta}{\omega_0} - n\theta\right)\delta(\mu - n\omega_0 - k_\parallel v_\parallel) \tag{39}$$

where the $\tilde{a}_n$ are subject only to the normalization condition

$$\int \mathrm{d}\vec{v}\,\eta(\vec{v})\tilde{G}_\mu(\vec{v}) = 1. \tag{40}$$

This means that the real eigenvalues of $\tilde{\mathcal{K}}$ are also infinitely degenerate. Therefore, we see that the adjoint operator $\tilde{\mathcal{K}}$ has the same spectrum structure and similar eigenfunctions as the original operator $\mathcal{K}$. By straightforward substitution, it can be proved that

$$\int \mathrm{d}\vec{v}\,[\tilde{\mathcal{K}} f(\vec{v})]^* g(\vec{v}) = \int \mathrm{d}\vec{v}\, f^*(\vec{v})[\mathcal{K} g(\vec{v})] \tag{41}$$

where '$*$' represents the complex conjugate. This relation leads easily to the orthogonal relation between the eigenfunctions of $\tilde{\mathcal{K}}$ and $\mathcal{K}$:

$$(z^* - z') \int \mathrm{d}\vec{v}\, \tilde{G}_z^*(\vec{v}) G_{z'}(\vec{v}) = 0. \tag{42}$$

One interesting point to note is that for the complex eigenvalues we have

$$\int \mathrm{d}\vec{v}\, \tilde{G}_{z_j}^*(\vec{v}) G_{z_j}(\vec{v}) = 0 \tag{43}$$

since $z_j^* \neq z_j$. The adjoint equation to (11) can be considered as a special case of $\vec{B}_0 = 0$; thus no additional treatment is necessary.

5. The resolvent method

As has been pointed out in the previous sections, the conventional eigenfunction expansion method is not suitable for solving equation (10). We therefore have to resort to other methods. It turns out that the resolvent method is a successful choice, just as it is for the one-dimensional equation (13) [6]. The resolvent of $\mathcal{K}$ is defined as

$$\mathcal{R}(z) = \frac{1}{z - \mathcal{K}} \tag{44}$$

where z is a complex variable. If the initial function is $f(\vec{v}, 0)$, then the function at time t is

$$f(\vec{v}, t) = \frac{1}{2\pi \mathrm{i}} \oint \mathrm{d}z\, \mathrm{e}^{-\mathrm{i}zt} \mathcal{R}(z) f(\vec{v}, 0) \tag{45}$$

where the integration contour surrounds all the singularities of $\mathcal{R}(z)$. For most operators, how their resolvents act on a function can be found only approximately. In this case, the explicit form of resolvent can be found exactly (the detailed derivation is given in the appendix). It is

$$\mathcal{R}(z) f(\vec{v}, 0) = F(z, \vec{v}) - G(z, \vec{v}) \frac{1}{\epsilon(z)} \int \mathrm{d}\vec{v}\, f(\vec{v}, 0) \tilde{G}^*(z^*, \vec{v}) \tag{46}$$

where

$$G(z, \vec{v}) = \exp\left(\mathrm{i} \frac{k_\perp v_\perp \sin\theta}{\omega_0}\right) \sum_n \mathrm{e}^{-\mathrm{i}n\theta} \frac{(J_{n+1} + J_{n-1})\eta_\perp/2 + J_n \eta_\parallel}{z - n\omega_0 - k_\parallel v_\parallel}$$

$$\tilde{G}(z, \vec{v}) = \exp\left(\mathrm{i} \frac{k_\perp v_\perp \sin\theta}{\omega_0}\right) \sum_n \mathrm{e}^{-\mathrm{i}n\theta} \frac{J_n}{z - n\omega_0 - k_\parallel v_\parallel}$$

$$F(z, \vec{v}) = \exp\left(\mathrm{i} \frac{k_\perp v_\perp \sin\theta}{\omega_0}\right) \sum_{m,n} f_m J_n \mathrm{e}^{-\mathrm{i}(m+n)\theta} \frac{1}{z - (m+n)\omega_0 - k_\parallel v_\parallel}$$

with

$$f(\vec{v}, 0) = \sum_m f_m(v_\parallel, v_\perp) \mathrm{e}^{\mathrm{i}m\theta}.$$

Similarly, for $\tilde{\mathcal{K}}$ we have

$$\frac{1}{z - \tilde{\mathcal{K}}} f(\vec{v}, 0) = F(z, \vec{v}) - \tilde{G}(z, \vec{v}) \frac{1}{\epsilon(z)} \int \mathrm{d}\vec{v}\, f(\vec{v}, 0) G^*(z^*, \vec{v}). \tag{47}$$

As is well known, the singularities of the resolvent $\mathcal{R}(z)$ give the spectrum of $\mathcal{K}$. It is not hard to see from (46) that the singularities of $\mathcal{R}(z)$ include the zeros of $\epsilon(z)$, the discrete real poles, $n\omega_0 + k_\parallel v_\parallel$, and a branch cut along the real axis. Therefore the spectrum of $\mathcal{K}$ is just

what we obtained earlier: all the real values and a set of discrete complex values determined by $\epsilon(z) = 0$. Let us write down the solution explicitly by choosing a specific contour:

$$f(\vec{v},t) = -\frac{1}{2\pi \mathrm{i}} \int_{C_+} \mathrm{d}z\, \mathrm{e}^{-\mathrm{i}zt} \left\{ F(z,\vec{v}) - G(z,\vec{v}) \frac{1}{\epsilon(z)} \int \mathrm{d}\vec{v}\, f(\vec{v},0) \tilde{G}^*(z^*,\vec{v}) \right\}$$
$$+ \frac{1}{2\pi \mathrm{i}} \int_{C_-} \mathrm{d}z\, \mathrm{e}^{-\mathrm{i}zt} \left\{ F(z,\vec{v}) - G(z,\vec{v}) \frac{1}{\epsilon(z)} \int \mathrm{d}\vec{v}\, f(\vec{v},0) \tilde{G}^*(z^*,\vec{v}) \right\} \tag{48}$$

where the $C_\pm$ run parallel to the real axis and are chosen such that all the singularities of $\mathcal{K}(z)$ are enclosed between them. We can use the residue theorem to calculate the above integration. If we are only interested in the solution for $t > 0$, then two large semi-circles can be attached from below to $C_\pm$ to make two closed contours $O_\pm$. Since there are no singularities inside the contour O_- the second part of the integration (48) is identical to zero. In contrast, O_+ encloses singularities: poles, such as z_j, and a branch cut along the real axis. To get rid of the troubling branch cut, we can replace the functions in the integrand with their plus analytic continuations [3]. These plus continuations are identical to their original functions in the upper complex half-plane but are different in the lower half-plane. They have no branch cuts but have new poles that differ from the original poles in the lower half-plane. These new poles are the mathematical origin of the Landau dampings [3, 4] in this plasma model.

When $\vec{B}_0 = 0$, equation (48) becomes

$$\frac{1}{z - \mathcal{K}_0} f(\vec{v},0) = \frac{f(\vec{v},0)}{z - \vec{k}\cdot\vec{v}} - \frac{\eta(\vec{v})}{z - \vec{k}\cdot\vec{v}} \frac{1}{\epsilon_0(z)} \int \mathrm{d}\vec{v}\, \frac{f(\vec{v},0)}{z - \vec{k}\cdot\vec{v}}. \tag{49}$$

Therefore, the solution of equation (11) is

$$f(\vec{v},t) = -\frac{1}{2\pi \mathrm{i}} \int_{C_+} \mathrm{d}z\, \mathrm{e}^{-\mathrm{i}zt} \left\{ \frac{f(\vec{v},0)}{z - \vec{k}\cdot\vec{v}} - \frac{\eta(\vec{v})}{z - \vec{k}\cdot\vec{v}} \frac{1}{\epsilon_0(z)} \int \mathrm{d}\vec{v}\, \frac{f(\vec{v},0)}{z - \vec{k}\cdot\vec{v}} \right\}$$
$$+ \frac{1}{2\pi \mathrm{i}} \int_{C_-} \mathrm{d}z\, \mathrm{e}^{-\mathrm{i}zt} \left\{ \frac{f(\vec{v},0)}{z - \vec{k}\cdot\vec{v}} - \frac{\eta(\vec{v})}{z - \vec{k}\cdot\vec{v}} \frac{1}{\epsilon_0(z)} \int \mathrm{d}\vec{v}\, \frac{f(\vec{v},0)}{z - \vec{k}\cdot\vec{v}} \right\}. \tag{50}$$

After the integration over the two components of $\vec{v}$ perpendicular to $\vec{k}$, the above equation can be reduced to one-dimensional form:

$$\bar{f}(v,t) = -\frac{1}{2\pi \mathrm{i}} \int_{C_+} \mathrm{d}z\, \mathrm{e}^{-\mathrm{i}zt} \left\{ \frac{\bar{f}(v,0)}{z - kv} - \frac{\bar{\eta}(v)}{z - kv} \frac{1}{\epsilon_0(z)} \int \mathrm{d}v\, \frac{\bar{f}(v,0)}{z - kv} \right\}$$
$$+ \frac{1}{2\pi \mathrm{i}} \int_{C_-} \mathrm{d}z\, \mathrm{e}^{-\mathrm{i}zt} \left\{ \frac{\bar{f}(v,0)}{z - kv} - \frac{\bar{\eta}(v)}{z - kv} \frac{1}{\epsilon_0(z)} \int \mathrm{d}v\, \frac{\bar{f}(v,0)}{z - kv} \right\} \tag{51}$$

which is the solution of equation (13).

It is important to note here that the eigenfunctions $\bar{g}_z(v)$ of $\bar{\mathcal{K}}_0$ can be constructed through its resolvent solution (51). The method was demonstrated clearly in [6]: the contour $C_\pm$ is divided into two parts, small circles surrounding the poles and two straight lines approaching to the real axis from both above and below. The first part straightforwardly gives the eigenfunctions $\bar{g}_i(v)$ corresponding to the complex eigenvalues. The second part leads to the real eigenfunctions $\bar{g}_\nu(v)$. For the operators $\mathcal{K}_0$ and $\mathcal{K}$, their complex eigenfunctions $g_j(\vec{v})$ and $G_j(\vec{v})$ can easily be constructed in a similar manner through their resolvent solutions (48) and (50), respectively. However, straightforward calculation immediately demonstrates the technical difficulties in constructing the real eigenfunctions $g_\mu(\vec{v})$ and $G_\mu(\vec{v})$ from (48) and (50) in like manner. It is not worth dwelling on the technical difficulties, which involve very complicated calculations and formulae, since one may argue that there may be another method by which these difficulties could be circumvented. Let us focus on the underlying reason, namely the uncountably infinite

degeneracy. Usually, the eigenfunctions of a linear operator can be labelled exactly either by a set of real numbers or integers or both. For the operators $\mathcal{K}_0$ and $\mathcal{K}$, I still labelled their eigenfunctions by real numbers and integers. However, the labelling is not exact since there are an infinite number of eigenfunctions for each real number. It certainly requires the introduction of non-trivial measure and other concepts in functional analysis to expand any function in the vector space expanded by the eigenfunctions $g_\mu(\vec{v})$ or $G_\mu(\vec{v})$. All of this is unlikely to be achieved by some clever algebraic manipulations with (48) and (50).

6. Conclusion

In summary, I have studied the linearized Vlasov equation of plasma in a constant external magnetic field $\vec{B}_0$ and found its exact solution by the resolvent method. The corresponding linear Vlasov operator $\mathcal{K}$ has also been discussed and its eigenfunctions found. The spectrum of this unbounded operator has two parts: continuous real eigenvalues and discrete complex eigenvalues. This is very similar to the spectrum of $\bar{\mathcal{K}}_0$, the one-dimensional linear operator derived from the linearized Vlasov equation without external fields [5, 7]. However, there is a striking new feature for $\mathcal{K}$ in that the real eigenvalues of $\mathcal{K}$ are uncountably infinitely degenerate. This new feature leads to the difficulty in expanding the functions in terms of the eigenfunctions of $\mathcal{K}$.

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Appendix

In this appendix I give the detailed derivation leading to (46). The eigenfunctions $G_j(\vec{v})$ and $G_\mu(\vec{v})$ can be found in a similar way. The method is called the integration along the characteristic curves [9] or the integration along the 'unperturbed orbits' [4]. In the following calculations, it is assumed that $\mathrm{Im}\, z > 0$. To make the formulae compact, let us use the notation:

$$F = (z - \mathcal{K}) f(\vec{v}) \tag{A1}$$

$$E = \exp(\mathrm{i}\vec{k} \cdot \vec{x} - \mathrm{i}zt). \tag{A2}$$

It is easy to verify with the aid of (9) that

$$\left[\mathrm{i}\frac{\partial}{\partial t} + \mathrm{i}\vec{v} \cdot \frac{\partial}{\partial \vec{x}} - \mathrm{i}\frac{e}{m_e}(\vec{v} \times \vec{B}_0) \cdot \frac{\partial}{\partial \vec{v}}\right](fE) = FE - E\eta \int f \tag{A3}$$

where $\eta = \eta(\vec{v})$, $f = f(\vec{v})$ and the non-specified integration is over the velocity. Along the 'unperturbed orbit', i.e. the orbit in the phase space of one electron moving in a magnetic field $\vec{B}_0$, we have

$$\frac{\mathrm{d}}{\mathrm{d}t} = \frac{\partial}{\partial t} + \vec{v} \cdot \frac{\partial}{\partial \vec{x}} - \frac{e}{m_e}(\vec{v} \times \vec{B}_0) \cdot \frac{\partial}{\partial \vec{v}}. \tag{A4}$$

Then equation (A3) becomes

$$\mathrm{i}\frac{\mathrm{d}}{\mathrm{d}t}(fE) = FE - \eta E \int f. \tag{A5}$$

Integrating this, we get

$$\mathrm{i}f = \int_{-\infty}^{t} \mathrm{d}t'\, F'E'E^{-1} - \left(\int_{-\infty}^{t} \mathrm{d}t'\, \eta' E' E^{-1}\right) \int f \tag{A6}$$

where the primed functions mean the functions of primed variables. Note that in the above integration over time t the fact $\mathrm{Im}\, z > 0$ has been used to choose $-\infty$ as the lower limit. A further integration with respect to velocity gives

$$\int f = \frac{\int\int_{-\infty}^{t} \mathrm{d}t'\, F'E'E^{-1}}{\mathrm{i} + \int\int_{-\infty}^{t} \mathrm{d}t'\, \eta' E' E^{-1}}. \tag{A7}$$

Plugging this back into (A6), we have

$$(z - \mathcal{K})^{-1} F = -\mathrm{i} \int_{-\infty}^{t} \mathrm{d}t'\, F'E'E^{-1} + \int_{-\infty}^{t} \mathrm{d}t'\, \eta' E' E^{-1} \frac{\int\int_{-\infty}^{t} \mathrm{d}t'\, F'E'E^{-1}}{\mathrm{i} + \int\int_{-\infty}^{t} \mathrm{d}t'\, \eta' E' E^{-1}}. \tag{A8}$$

Adopting the coordinate system and notation in (27) and (28), we write out (A8) explicitly:

$$\begin{aligned} \mathcal{R}(z) f(\vec{v}) = {} & -\mathrm{i} \int_0^{\infty} \mathrm{d}\tau\, f(v_\parallel, v_\perp, \theta - \omega_0 t) \mathrm{e}^{-\mathrm{i}\phi(\tau)} \\ & + \int_0^{\infty} \mathrm{d}\tau\, \eta(\vec{v}') \mathrm{e}^{-\mathrm{i}\phi(\tau)} \frac{\int \mathrm{d}\vec{v} \int_0^{\infty} \mathrm{d}\tau\, f(v_\parallel, v_\perp, \theta - \omega_0 t) \mathrm{e}^{-\mathrm{i}\phi(\tau)}}{1 - \mathrm{i} \int \mathrm{d}\vec{v} \int_0^{\infty} \mathrm{d}\tau\, \eta(\vec{v}') \mathrm{e}^{-\mathrm{i}\phi(\tau)}} \end{aligned} \tag{A9}$$

where

$$\phi(\tau) = \frac{k_\perp v_\perp}{\omega_0} (\sin\theta - \sin(\theta - \omega_0 \tau)) + (k_\parallel v_\parallel - z)\tau \tag{A10}$$

$$\eta(\vec{v}') = \eta_\perp \cos(\theta - \omega_0 \tau) + \eta_\parallel. \tag{A11}$$

The case $\mathrm{Im}\, z < 0$ can be dealt with similarly, which also leads to (A9). The step from (A9) to (46) is direct integration with respect to τ using Bessel functions.

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