New Journal of Physics
The open access journal at the forefront of physics
Deutsche Physikalische Gesellschaft DPG IOP Institute of Physics

PAPER • OPEN ACCESS

Lorentz quantum mechanics

To cite this article: Qi Zhang and Biao Wu 2018 *New J. Phys.* **20** 013024

IOP Publishing New J. Phys. 20 (2018) 013024 https://doi.org/10.1088/1367-2630/aa8496

New Journal of Physics
The open access journal at the forefront of physics

Deutsche Physikalische Gesellschaft DPG
IOP Institute of Physics
Published in partnership with: Deutsche Physikalische Gesellschaft and the Institute of Physics

OPEN ACCESS

RECEIVED
30 May 2017

REVISED
26 July 2017

ACCEPTED FOR PUBLICATION
7 August 2017

PUBLISHED
22 January 2018

PAPER

Lorentz quantum mechanics

Qi Zhang (张起)[1] and Biao Wu (吴飙)[2,3,4]

[1] College of Science, Zhejiang University of Technology, Hangzhou 310023, People's Republic of China
[2] International Center for Quantum Materials, School of Physics, Peking University, Beijing 100871, People's Republic of China
[3] Collaborative Innovation Center of Quantum Matter, Beijing 100871, People's Republic of China
[4] Wilczek Quantum Center, School of Physics and Astronomy, Shanghai Jiao Tong University, Shanghai 200240, People's Republic of China

Keywords: Lorentz transformation, Berry phase, bosonic Bogoliubov quasiparticles, complex Minkowski space, PT-symmetric quantum system

Abstract

We present a theoretical framework for the dynamics of bosonic Bogoliubov quasiparticles. We call it Lorentz quantum mechanics because the dynamics is a continuous complex Lorentz transformation in complex Minkowski space. In contrast, in usual quantum mechanics, the dynamics is the unitary transformation in Hilbert space. In our Lorentz quantum mechanics, three types of state exist: space-like, light-like and time-like. Fundamental aspects are explored in parallel to the usual quantum mechanics, such as a matrix form of a Lorentz transformation, and the construction of Pauli-like matrices for spinors. We also investigate the adiabatic evolution in these mechanics, as well as the associated Berry curvature and Chern number. Three typical physical systems, where bosonic Bogoliubov quasi-particles and their Lorentz quantum dynamics can arise, are presented. They are a one-dimensional fermion gas, Bose–Einstein condensate (or superfluid), and one-dimensional antiferromagnet.

1. Introduction

Bosonic Bogoliubov quasiparticles arise in many different physical systems [1, 2]. They have been studied extensively in condensed matter physics for their static properties, such as dispersion, and in particular, their relation to superfluidity [3–5]. Inspired partly by the work in [6], where the dynamics of Bogoliubov quasiparticles in a superfluid with a vortex is studied, we present here a general theoretical framework for such dynamics. As the bosonic Bogoliubov operator is non-Hermitian, we find that the dynamics is a continuous Lorentz transformation of a state in complex Minkowski space. In contrast, the usual quantum dynamics is a continuous unitary transformation of a state in Hilbert space. For this reason, we call the dynamics of bosonic Bogoliubov quasiparticles Lorentz quantum mechanics.

In Lorentz quantum mechanics, we find that the interval of a state is conserved and therefore the complex Minkowski space has three subspaces: space-like, light-like, and time-like, which are invariant during the dynamic evolution. In this work we focus on the $(1,\ 1)$-type spinor, the simplest Lorentz spinor, and use this example to explore in which ways the Lorentz quantum mechanics are similar to, and different from, the conventional quantum mechanics. In particular, we construct the matrix representing the Lorentz transformation of complex vectors, and the Lorentz counterpart of the standard Pauli matrices. The Berry phase is also investigated in the context of Lorentz quantum mechanics and it is found to be quite different from the Berry phase in usual quantum mechanics.

In the end, we give three specific physical systems: the spin wave excitations in a one-dimensional (1D) antiferromagnetic system, the phonon excitations on top of a vortex in the Bose–Einstein condensate (BEC), and a 1D fermion gas at low temperatures, where Lorentz quantum mechanics can arise. We use these systems to further illustrate our general results. In particular, with the antiferromagnetic system we point out explicitly how spin–orbit coupling can arise in Lorentz quantum mechanics.

We note that the non-Hermitian Hamiltonian has been extensively studied in the context of PT-symmetric quantum mechanics, where the spectrum (eigenvalue) of non-Hermitian operator is proved to be real [7]. The

PT-symmetric structure has found extensive applications in phonon-laser (coupled-resonator) systems, where giant nonlinearity arises in the vicinity of phase transition between PT-symmetric phase and broken-PT phase, resulting in enhanced mechanical sensitivity [8], optical intensity [9], controllable chaos [10] and optomechanically-induced transparency [11], as well as the phonon-rachet effect [12]. The geometric phase of PT-symmetric quantum mechanics [13] and the stability of driving non-Hermitian system has also been studied [14]. The bosonic Bogoliubov operator studied here stands for a class of generalized PT-symmetric Hamiltonian [15], or more precisely, the anti-PT Hamiltonian [16], which can be realized experimentally by making use of refractive indices in optical settings [16, 17]. It will be interesting to examine the general theoretical framework of these PT-symmetric quantum mechanics in the future.

2. Basic structures of Lorentz quantum mechanics

The Lorentz quantum mechanics is described by the following dynamical equation

$$i\hbar\frac{\mathrm{d}}{\mathrm{d}t}\begin{pmatrix} a_1(t) \\ a_2(t) \\ \vdots \\ a_{m+n}(t) \end{pmatrix} = \sigma_{m,n}H\begin{pmatrix} a_1(t) \\ a_2(t) \\ \vdots \\ a_{m+n}(t) \end{pmatrix}, \tag{1}$$

where $H = H^\dagger$ is a Hermitian matrix while $\sigma_{m,n}$ is given by

$$\sigma_{m,n} = \mathrm{diag}\{\underbrace{1,\ 1,\dots 1}_{m},\ \underbrace{-1,\ -1,\dots -1}_{n}\}. \tag{2}$$

Equations of this type are usually called Bogoliubov-de Gennes (BdG) equations and are obeyed by bosonic quasi-particles in many different physical systems (see section 4). For simplicity, we use the case $\sigma_{1,1}$ to explore the basic structures of the Lorentz quantum mechanics as generalization to $\sigma_{m,n}$ is straightforward.

2.1. Complex Lorentz transformation and complex Minkowski space

The BdG equation for spinor (1, 1) is

$$i\hbar\frac{\mathrm{d}}{\mathrm{d}t}\begin{pmatrix} a(t) \\ b(t) \end{pmatrix} = \sigma_{1,1}H\begin{pmatrix} a(t) \\ b(t) \end{pmatrix}. \tag{3}$$

Here $a(t)$ and $b(t)$ are the standard bosonic Bogoliubov amplitudes, $H = H^\dagger$ is a Hermitian matrix, and $\sigma_{1,1} = \sigma_z$ is the familiar Pauli matrix in the z direction, i.e.

$$\sigma_{1,1} = \mathrm{diag}\{1,\ -1\} = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}. \tag{4}$$

The $\sigma_{1,1}H$ as the generator of the dynamics for spinor $(1,\ 1)$ is an analogue of the Hamiltonian in the Schrödinger picture. Different from the Hamiltonian, though, $\sigma_{1,1}H$ is not Hermitian; it generates complex Lorentz transformation in complex Minkowski space as we shall see.

For an arbitrary initial state $|\psi(0)\rangle = [a(0),\ b(0)]^T$, the wavefunction $|\psi(t)\rangle = [a(t),\ b(t)]^T$ at times $t > 0$ can be solved formally from equation (3) as

$$|\psi(t)\rangle = \mathcal{U}(t,\ 0)|\psi(0)\rangle. \tag{5}$$

Here $\mathcal{U}(t,\ 0)$ is the evolution operator defined by

$$\mathcal{U}(t,\ 0) = \mathrm{e}^{-i\sigma_{1,1}Ht/\hbar}. \tag{6}$$

The goal of this section is to show that the operator $\mathcal{U}(t,\ 0)$ defined in equation (6) generates a *complex Lorentz*—instead of a unitary-evolution of $|\psi(t)\rangle$. In particular, defining the interval for a Lorentz spinor

$$\mathrm{In}((a,\ b)^T) = (a^*,\ b^*)\sigma_{1,1}(a,\ b)^T = |a|^2 - |b|^2, \tag{7}$$

we prove below that the interval is conserved under the evolution generated by $\mathcal{U}(t,\ 0)$, i.e.

$$|a(t)|^2 - |b(t)|^2 = |a(0)|^2 - |b(0)|^2. \tag{8}$$

For the above purpose, we first establish the following relation

$$\mathcal{U}^\dagger\sigma_{1,1}\mathcal{U} = \sigma_{1,1}. \tag{9}$$

Expanding $\sigma_{1,1}\mathcal{U}$ and $(\mathcal{U}^\dagger)^{-1}\sigma_{1,1}$ in Taylor series, and noting $\sigma_{1,1}\sigma_{1,1} = 1$, the nth term in the expansions of both $\sigma_{1,1}\mathcal{U}$ and $(\mathcal{U}^\dagger)^{-1}\sigma_{1,1}$ are of the form

$$\frac{1}{n!}\left(-\frac{\mathrm{i}}{\hbar}\right)^n t^n H \underbrace{\sigma_{1,1}H\sigma_{1,1}H\ \ldots\ \sigma_{1,1}H}_{n-1(\sigma_{1,1}H)\mathrm{s}}. \tag{10}$$

This readily gives

$$\sigma_{1,1}\mathcal{U} = (\mathcal{U}^{\dagger})^{-1}\sigma_{1,1}, \tag{11}$$

from which equation (9) ensues. Hence, by virtue of equation (9), we obtain

$$\langle\psi(t)|\sigma_{1,1}|\psi(t)\rangle = \langle\psi(0)|\sigma_{1,1}|\psi(0)\rangle, \tag{12}$$

and thus equation (8). Similarly, we can show that $\mathcal{U}^{\dagger}\mathcal{U} \neq 1$ and $\langle\psi(t)|\psi(t)\rangle$ is not conserved during the dynamical evolution.

It is clear from the above results that the vector space spanned by states $|\psi(t)\rangle$ is not a Hilbert space and the evolution operator $\mathcal{U}$ is not a unitary transformation. Due to its mathematical similarity to the Lorentz transformation in Minkowski space, we call the vector space spanned by states $|\psi(t)\rangle$ complex Minkowski space and any operator satisfying equation (9) Lorentz transformation. For the $(1,\ 1)$-spinor, the general matrix form of the Lorentz transformation is

$$\mathfrak{L} = \begin{pmatrix} \zeta & \eta^* \\ \eta & \zeta^* \end{pmatrix}, \tag{13}$$

where $|\zeta|^2 - |\eta|^2 = 1$, with the corresponding inverse Lorentz matrix being

$$\mathfrak{L}^{-1} = \begin{pmatrix} \zeta^* & -\eta^* \\ -\eta & \zeta \end{pmatrix}. \tag{14}$$

To avoid any confusion, we reiterate that our Lorentz transformation is a mathematical generalization of the Lorentz transformation of special relativity to complex numbers. Physically, they are very different; our Lorentz transformation operates on states of bosonic Bogoliubov quasiparticles which form a complex Minkowski space; the Lorentz transformation in special relativity operates on space-time which is a real Minkowski space.

As the interval defined in equation (8) does not change under Lorentz transformation, the complex Minkowski space where the states of bosonic Bogoliubov quasiparticles reside in has three subspaces up to the normalization constant, which are defined by $\mathrm{In}((a,\ b)^T) = |a|^2 - |b|^2 > 0$, $\mathrm{In}((a,\ b)^T) = |a|^2 - |b|^2 = 0$ and $\mathrm{In}((a,\ b)^T) = |a|^2 - |b|^2 < 0$. To set the convention, we call them space-like, light-like, and time-like, respectively. Physically, if the space-like states with $|a|^2 - |b|^2 > 0$ describe bosonic Bogoliubov quasiparticles, then the time-like states with $|a|^2 - |b|^2 < 0$ describe the corresponding anti-particles [5].

We thus conclude that the dynamical evolution generated by $\mathcal{U}(t,\ 0)$ conserves the interval (see equation (8)), and therefore, is a continuous *complex Lorentz* transformation in complex Minkowski space.

Before we proceed to explore other properties of this Lorentz quantum dynamics, we take a sidestep to point out that the BdG equation is a special class of PT-symmetric quantum mechanics [15, 18–21]. The general form of two-mode PT-symmetric Hamiltonian has been written as [15, 18–21]

$$H_{\mathrm{PT}} = \begin{pmatrix} \epsilon + \gamma\cos\theta - \mathrm{i}\mu\sin\theta & (\gamma\sin\theta + \mathrm{i}\mu\cos\theta + \nu)\mathrm{e}^{-\mathrm{i}\varphi} \\ (\gamma\sin\theta + \mathrm{i}\mu\cos\theta - \nu)\mathrm{e}^{\mathrm{i}\varphi} & \epsilon - \gamma\cos\theta + \mathrm{i}\mu\sin\theta \end{pmatrix}, \tag{15}$$

where $\epsilon, \mu, \nu, \gamma, \theta$ and φ are real parameters. In fact, both Hermitian Hamiltonian H and BdG Hamiltonian $\sigma_{1,1}H$ are special cases of the PT-symmetric Hamiltonian (15). It follows from (15) that the two-mode BdG Hamiltonian $\sigma_{1,1}H$ recovers from H_{PT} when $\epsilon = 0$, $\theta = 2N\pi$ and $\varphi = 2N\pi$; while the Hermitian Hamiltonian H recovers when $\mu = \nu = 0$.

2.2. Eigen-energies and eigenstates

Although the $\sigma_{1,1}H$ is not Hermitian, under certain conditions, it can admit real eigenvalues—which are relevant for physical processes. We write $\sigma_{1,1}H$ in terms of three basic matrices as (dropping the term involving the identity matrix)

$$\sigma_{1,1}H = m_1\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} + m_2\begin{pmatrix} 0 & \mathrm{i} \\ \mathrm{i} & 0 \end{pmatrix} + m_3\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \tag{16}$$

where the parameters m_i $(i = 1,\ 2,\ 3)$ are real. The eigen-energies are the roots of the following equation

$$m_3^2 - (m_1^2 + m_2^2) = E^2. \tag{17}$$

It is clear that the eigenvalues are real provided the condition

$$m_3^2 \geqslant m_1^2 + m_2^2 \tag{18}$$

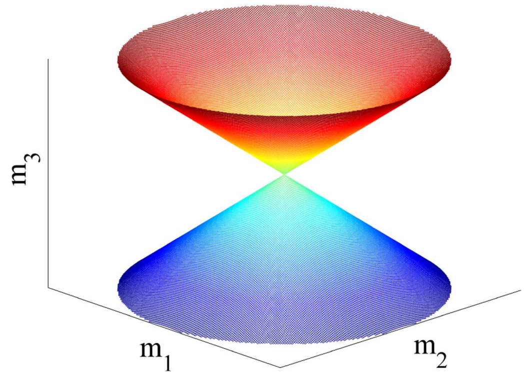

Figure 1. The degeneracy regime of Lorentz spinor parameterized by m_1, m_2 and m_3 as in equation (16) forms the surface of a cone. As will be discussed in section 3, the charge (monopole) for the Berry curvature (monopole) is at the tip of the cone rather than distributing over the whole degeneracy cone.

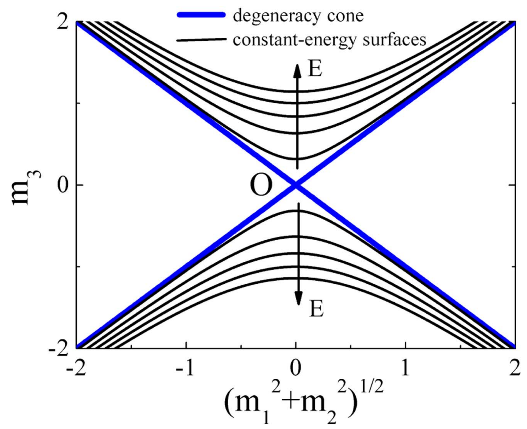

Figure 2. Illustration of the constant-energy surfaces of BdG equation parameterized by m_1, m_2 and m_3. The arrows indicate the directions of increasing (decreasing) of energy for state $|1\rangle$ ($|2\rangle$). On the cone's surface, the two eigenstates are degenerate. Because the surfaces assume the axial symmetry about the m_3 axis, the two-dimensional plot is depicted for clarity.

is satisfied. In this work, we shall restrict ourselves to this physically relevant regime of real-eigenvalues in the parameter domain specified by $(m_1,\ m_2,\ m_3)$, and we denote the two real eigenvalues as E_1 and E_2, with the corresponding eigenstates labeled as $|1\rangle$ and $|2\rangle$, respectively.

Two facts are clear from equation (17): (i) in the parameter space $(m_1,\ m_2,\ m_3)$, the two eigenstates $|1\rangle$ and $|2\rangle$ exhibit degeneracies on a circular cone (see figure 1), which resembles the light-cone in special relativity. This is in marked contrast to a unitary spinor, where the degeneracy occurs only at an isolated point; (ii) unlike a unitary spinor where the constant-energy surfaces are elliptic surfaces, both eigenstates of $\sigma_{1,1}H$ display hyperbolic constant-energy surfaces (see figure 2).

We now describe the basic properties of the eigenstates associated with the operator $\sigma_{1,1}H$. They are solutions to the following eigen-equations

$$\sigma_{1,1}H|1\rangle = E_1|1\rangle, \tag{19}$$

$$\sigma_{1,1}H|2\rangle = E_2|2\rangle. \tag{20}$$

Keeping in mind that only real eigenvalues are considered, for $E_1 \neq E_2^* = E_2$, we have

$$\langle 2|\sigma_{1,1}|1\rangle = 0. \tag{21}$$

It can be checked that the two eigenstates of $\sigma_{1,1}H$ can always be specifically expressed as

$$|1\rangle = \begin{pmatrix} u \\ v \end{pmatrix}; \quad |2\rangle = \begin{pmatrix} v^* \\ u^* \end{pmatrix}. \tag{22}$$

This means that if $|1\rangle$ is space-like then $|2\rangle$ is time-like or vice versa.

In the energy representation defined in terms of $|1\rangle$ and $|2\rangle$, a time-evolved state $|\psi(t)\rangle = [a(t),\ b(t)]^T$ (see equation (3)) can be written as

$$|\psi(t)\rangle = c_1|1\rangle e^{-iE_1 t} + c_2|2\rangle e^{-iE_2 t}. \tag{23}$$

In transforming $|\psi(t)\rangle$ from the Bogoliubov representation to the energy representation, the interval of the Lorentz spinor is preserved, i.e. it is a complex Lorentz transformation. To see this, using equation (21), we find

$$\begin{aligned} \mathrm{In}(|\psi(t)\rangle) &= \langle\psi(t)|\sigma_{1,1}|\psi(t)\rangle \\ &= |c_1|^2\langle 1|\sigma_{1,1}|1\rangle + |c_2|^2\langle 2|\sigma_{1,1}|2\rangle. \end{aligned} \tag{24}$$

By further assuming a gauge for Lorentz-like normalization, i.e.

$$\begin{aligned} \mathrm{In}(|1\rangle) &= \langle 1|\sigma_{1,1}|1\rangle = 1, \\ \mathrm{In}(|2\rangle) &= \langle 2|\sigma_{1,1}|2\rangle = -1, \end{aligned} \tag{25}$$

we obtain from (24) that

$$\mathrm{In}(|\psi(t)\rangle) = |a|^2 - |b|^2 = |c_1|^2 - |c_2|^2, \tag{26}$$

meaning the interval is conserved for the above representation transformation.

The normalization condition $|u|^2 - |v|^2 = 1$ is different from the eigenstates of a conventional unitary spinor. In fact, if one naively enforce the unitary gauge on equation (22), say, $|u|^2 + |v|^2 = 1$, *unphysical* consequences would ensue; the time-evolved wavefunction in the original Bogoliubov representation [$|\psi\rangle = (a,\ b)^T$] could not maintain its ordinary amplitude, such that $|a(t)|^2 + |b(t)|^2 \neq 1$ for $t > 0$, and, in particular, the amplitude in different representation would take different value, e.g. $|c_1|^2 + |c_2|^2 \neq |a(t)|^2 + |b(t)|^2$, which can be easily inferred from equation (23).

In general, when $\sigma_{1,1}H$ takes the form (16) with $m_3 = 0$, it exhibits two light-like eigenvectors; whereas, when $m_3 \neq 0$, there are one space-like and one time-like eigenvectors. Thus, in the physically relevant regime $m_3^2 \geqslant m_1^2 + m_2^2$ as considered here, we find $|1\rangle$ is space-like and $|2\rangle$ time-like. As a result, a light-like vector can be formed from a superposition of two eigenvectors with equal weight, i.e. $|c_1|^2 - |c_2|^2 = |a|^2 - |b|^2 = 0$.

2.3. Representation transformation and physical meaning of the wavefunction

In quantum mechanics, the change from one representation to another (or from one basis to another) is given by a unitary matrix. In Lorentz quantum mechanics, the representation transformation

$$|\psi\rangle \to |\psi'\rangle = \mathfrak{L}|\psi\rangle \tag{27}$$

is facilitated by Lorentz matrix $\mathfrak{L}$ in equation (13). Correspondingly, an operator K transforms as

$$K \to K' = \mathfrak{L}K\mathfrak{L}^{-1}. \tag{28}$$

Note that, since $\mathfrak{L}$ is not a unitary matrix, we have $\mathfrak{L}^{-1} \neq \mathfrak{L}^\dagger$.

As an example, we consider the transformation from the Bogoliubov representation to the energy representation as described earlier. In this case, the eigenstates $|1\rangle$ and $|2\rangle$ transform as

$$|1\rangle = \begin{pmatrix} u \\ v \end{pmatrix} \to \mathfrak{L}_{\mathrm{B}}|1\rangle = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \tag{29}$$

$$|2\rangle = \begin{pmatrix} v^* \\ u^* \end{pmatrix} \to \mathfrak{L}_{\mathrm{B}}|2\rangle = \begin{pmatrix} 0 \\ 1 \end{pmatrix}, \tag{30}$$

where the matrix $\mathfrak{L}_{\mathrm{B}}$ is shown in equation (13), with $\zeta = u^*$ and $\eta = -v$, i.e.

$$\mathfrak{L}_{\mathrm{B}} = \begin{pmatrix} u^* & -v^* \\ -v & u \end{pmatrix}. \tag{31}$$

Obviously, as we have proven, the interval must be conserved, i.e., $|u|^2 - |v|^2 = 1^2 - 0^2 = 1$, $|v^*|^2 - |u^*|^2 = 0^2 - 1^2 = -1$. In addition, the bosonic Bogoliubov operator transforms as,

$$\sigma_{1,1}H \to \mathfrak{L}_{\mathrm{B}}\sigma_{1,1}H\mathfrak{L}_{\mathrm{B}}^{-1} = \begin{pmatrix} E_1 & 0 \\ 0 & E_2 \end{pmatrix}. \tag{32}$$

This special Lorentz transformation from the original representation to the energy representation is just the well-known Bogoliubov transformation for bosons [5, 22, 23].

In light of the conservation of interval—rather than norm—of the state vector under transformations, a question immediately arises as to whether, or to what extent, the wavefunction in the context of Lorentz quantum mechanics still affords the physical interpretation as the probability wave? Indeed, in the energy representation, see equation (23), it is clear that $|c_{1(2)}|^2$, with $|c_1|^2 + |c_2|^2 = 1$, can be interpreted as the probability of finding the spinor in the eigenstate $|1(2)\rangle$, i.e., a wavefunction $c_1|1\rangle + c_2|2\rangle$ still describes a probability wave. However, in the Bogoliubov representation, the interpretation of a wavefunction as the probability wave is no longer physically meaningful. For example, consider the eigenstate $|1\rangle = (u,\ v)^T$, which is usually generated from creating a pair of Bogoliubov quasiparticles in the ground state of the system. Yet, $|u|^2$

and $|v|^2$ cannot represent the probabilities in the Bogoliubov basis; the Bogoliubov basis is not a set of orthonormal basis (see section 4 for concrete examples), and therefore, instead of $|u|^2 + |v|^2 = 1$, the convention $|u|^2 - |v|^2 = 1$ must be taken.

2.4. Completeness of eigenvectors

Based on equation (22) (see also equations (21) and (25)), the completeness of eigenvectors in the energy representation now takes a different form compared to the unitary case, reading

$$\bigsqcup_j |j\rangle\langle j|\sigma_{1,1} = 1, \tag{33}$$

or, equivalently,

$$\sigma_{1,1}\bigsqcup_j |j\rangle\langle j| = 1. \tag{34}$$

Here, the notation $\bigsqcup_j$ [for $(1+1)$-mode] is defined by

$$\bigsqcup_j |j\rangle\langle j| = |1\rangle\langle 1| - |2\rangle\langle 2|. \tag{35}$$

It can easily be found that, ensured by the property of Lorentz matrix $\mathfrak{L}^\dagger\sigma_{1,1}\mathfrak{L} = \sigma_{1,1}$, the completeness expression (33) (or (34)) remains in any other representation.

2.5. Analogue of Pauli matrices

In analogy with the conventional spinor that is acted by the basic operators known as Pauli matrices, it is natural to ask, for the Lorentz spinor, if similar matrices can be constructed. Such analogue of the Pauli matrices, denoted by τ_i $(i = 1, 2, 3)$, is required to fulfill the following conditions: (i) any operator $\sigma_{1,1}H$, when written in terms of τ_i (dropping the term involving identity matrix), i.e.

$$\sigma_{1,1}H = n_1\tau_1 + n_2\tau_2 + n_3\tau_3, \tag{36}$$

must have real-number components n_i; (ii) the matrices τ_i $(i = 1, 2, 3)$ should have the same real eigenvalues, say, ± 1, and can transform into each other via Lorentz transformation (see equation (28)).

Based on (i) and (ii), we see that the matrices as appeared in equation (16) do not represent the analogue of the Pauli matrix for the Lorentz spinor: while they satisfy the requirement (i), the condition (ii) is violated. Instead, we consider the following constructions:

$$\tau_1 = \begin{pmatrix} \sqrt{2} & 1 \\ -1 & -\sqrt{2} \end{pmatrix},\quad \tau_2 = \begin{pmatrix} \sqrt{2} & \mathrm{i} \\ \mathrm{i} & -\sqrt{2} \end{pmatrix},\quad \tau_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}. \tag{37}$$

It is easy to check that τ_i in equation (37) satisfy both requirements (i) and (ii). In particular, the transformation between τ_1 and τ_3 is explicitly found to be

$$\tau_1 = \mathfrak{L}\tau_3\mathfrak{L}^{-1}, \tag{38}$$

where $\mathfrak{L}$ is of the form (13) with $\zeta = \frac{\sqrt{2}+1}{2}\mathrm{i}$ and $\eta = -\frac{\sqrt{2}-1}{2}\mathrm{i}$, and that between τ_2 and τ_3 is given by

$$\tau_2 = \mathfrak{L}\tau_3\mathfrak{L}^{-1}, \tag{39}$$

for $\mathfrak{L}$ with $\zeta = \frac{\sqrt{2}+1}{2}\mathrm{e}^{-\mathrm{i}\frac{\pi}{4}}$ and $\eta = -\frac{\sqrt{2}-1}{2}\mathrm{e}^{\mathrm{i}\frac{\pi}{4}}$.

2.6. Heisenberg picture

The current Lorentz evolution is in fact defined in the analogue of Schrödinger picture (denoted by subscript s), i.e. any physical operator keeps constant while the wavefunction undergoes Lorentz evolution. In analogy with the conventional spinor, the Lorentz quantum mechanics can also be expressed in the analogue of Heisenberg picture (denoted by subscript h). The relations of an operator $\mathcal{O}$ and the state $|\psi\rangle$ between the two pictures are

$$\mathcal{O}(t)_h = \mathrm{e}^{\mathrm{i}\sigma_{1,1}Ht}\mathcal{O}_s\mathrm{e}^{-\mathrm{i}\sigma_{1,1}Ht}, \tag{40}$$

$$|\psi\rangle_h = \mathrm{e}^{\mathrm{i}\sigma_{1,1}Ht}|\psi(t)\rangle_s, \tag{41}$$

where $|\psi\rangle_h$ keeps constant but $\mathcal{O}(t)_h$ satisfies the analogue of Heisenberg equation,

$$\mathrm{i}\hbar\frac{\partial\mathcal{O}(t)_h}{\partial t} = [\sigma_{1,1}H, \mathcal{O}(t)_h], \tag{42}$$

with $[\sigma_{1,1}H, \mathcal{O}(t)_h]$ being the commutator between $\sigma_{1,1}H$ and $\mathcal{O}(t)_h$.

2.7. Generalization to multi-mode

In this section, we extend the above formulations for the $\sigma_{1,1}$ Lorentz spinor to the case of multi-mode spinor with $\sigma_{m,n}$. The operator $\sigma_{m,n}H$ has $m+n$ energy eigenstates, denoted by $|1\rangle, |2\rangle, \ldots, |m+n\rangle$. Defining the interval of a $(m+n)$-mode wavefunction $|\psi\rangle = (a_1, a_2, \ldots, a_{m+n})^T$ as

$$\mathrm{In}(|\psi\rangle) = \langle\psi|\sigma_{m,n}|\psi\rangle = \sum_{j=1}^{m}|a_j|^2 - \sum_{j=m+1}^{m+n}|a_j|^2. \tag{43}$$

It is easy to see that the intervals of the eigenstates are

$$\begin{aligned} &\mathrm{In}(|j\rangle) = 1 \quad \text{for} \quad j = 1, 2, \ldots m, \\ &\mathrm{In}(|j\rangle) = -1 \quad \text{for} \quad j = m+1, m+2, \ldots m+n. \end{aligned} \tag{44}$$

In addition, the orthogonal condition for two non-degenerate eigenstates is derived as

$$\langle j|\sigma_{m,n}|k\rangle = 0, \quad \text{for} \quad j \neq k, \tag{45}$$

generalizing equation (21) for the $(1,\ 1)$-mode. Using equations (44) and (45), the completeness of eigenvectors can be expressed as

$$\bigsqcup_j |j\rangle\langle j|\sigma_{m,n} = 1, \tag{46}$$

or, equivalently

$$\sigma_{m,n}\bigsqcup_j |j\rangle\langle j| = 1, \tag{47}$$

with the symbol $\bigsqcup_j$ for (m, n)-mode defined as

$$\bigsqcup_j |j\rangle\langle j| = \sum_{j=1}^{m}|j\rangle\langle j| - \sum_{j=m+1}^{m+n}|j\rangle\langle j|. \tag{48}$$

3. Adiabaticity and geometric phase

3.1. Adiabatic theorem

Consider a $(1,\ 1)$-spinor described by the operator $\sigma_{1,1}H(\mathbf{R})$, which depends on a set of system's parameter $\mathbf{R}$. Suppose the spinor is initially in an eigenstate, say $|1\rangle$, before the parameter $\mathbf{R}$ undergoes a sufficiently slow variation, thus driving an adiabatic evolution for the Lorentz spinor. The relevant matrix element capturing the slowly varying time-dependent perturbation can be evaluated as, by acting the gradient operator $\nabla \equiv \frac{\partial}{\partial\mathbf{R}}$ on the equation (19) and using equation (20),

$$\langle 2|\sigma_{1,1}\nabla|1\rangle = \frac{\langle 2|\nabla H|1\rangle}{E_1 - E_2^*} = \frac{\langle 2|\nabla H|1\rangle}{E_1 - E_2}. \tag{49}$$

Here, the last equality is ensured by the real eigenvalues in the considered parameter regimes, together with the condition $E_1 \neq E_2$.

We see that the relation (49), except for an additional $\sigma_{1,1}$, is identical with that in unitary quantum mechanics [24]. This allows us to generalize the familiar adiabatic theorem to the context of Lorentz quantum mechanics; starting from an initial eigenstate $|1(\mathbf{R})\rangle$ ($|2(\mathbf{R})\rangle$), the system will always be constrained in this instantaneous eigenstate so long as $\mathbf{R}$ is swept slowly enough in the parameter space. (A rigorous proof would be similar to that in the conventional quantum mechanics [24, 25], and therefore, here we shall leave out the detailed procedure.)

3.2. Analogue of Berry phase

In conventional quantum mechanics, it is well-known that an eigen-energy state undergoing an adiabatic evolution will pick up a Berry phase [26], when a slowly varying system parameter $\mathbf{R}$ realizes a loop in the parameter space. Here we show that in the context of Lorentz quantum mechanics, a Lorentz counterpart of the Berry phase will similarly arise.

The time evolution of an instantaneous eigenstate, which is parametrically dependent on $\mathbf{R}$, can be written as

$$|\psi\rangle = |m\rangle\,\mathrm{e}^{-\mathrm{i}\frac{\int E_m(\mathbf{R})\mathrm{d}t}{\hbar}}\mathrm{e}^{\mathrm{i}\beta}, \tag{50}$$

with $m = 1, 2$. Here, $-\int E_m(\mathbf{R})\mathrm{d}t/\hbar$ denotes the dynamical phase and β the geometric phase. Substituting equation (50) into equation (3), we find

$$\frac{\mathrm{d}\beta_1}{\mathrm{d}\mathbf{R}} = \mathrm{i}\langle 1|\sigma_{1,1}\frac{\partial}{\partial \mathbf{R}}|1\rangle; \tag{51}$$

and

$$\frac{\mathrm{d}\beta_2}{\mathrm{d}\mathbf{R}} = -\mathrm{i}\langle 2|\sigma_{1,1}\frac{\partial}{\partial \mathbf{R}}|2\rangle. \tag{52}$$

From equations (51) and (52), we can readily read off the Berry connections as

$$\mathbf{A}_1 = \mathrm{i}\langle 1|\sigma_{1,1}\nabla|1\rangle, \tag{53}$$

$$\mathbf{A}_2 = -\mathrm{i}\langle 2|\sigma_{1,1}\nabla|2\rangle. \tag{54}$$

Equations (53) and (54) show that the Berry connection in the Lorentz quantum mechanics is modified from the conventional one, where the Berry connection is given by $\mathrm{i}\langle m|\frac{\partial}{\partial \mathbf{R}}|m\rangle$. Will such modifications give rise to a different monopole structure for the Berry curvature? Or, will the monopole in the Lorentz mechanics still occur at the degeneracy point (where $E_1 = E_2$)? To address these questions, we now calculate the Berry curvature $\mathbf{B} = \nabla \times \mathbf{A}$. Without loss of generality, we take the eigenvector $|1\rangle$ for concrete calculations.

Our starting point is the identity $\langle 1|\sigma_{1,1}|1\rangle = 1$. By acting ∇ on both sides, we obtain

$$\langle 1|\sigma_{1,1}\nabla|1\rangle + \langle 1|\sigma_{1,1}\nabla|1\rangle^* = 0. \tag{55}$$

This indicates that $\langle 1|\sigma_{1,1}\nabla|1\rangle$ is purely imaginary ($\mathbf{A}_1$ is real). Hence, $\mathbf{B}_1$ can be evaluated as

$$\mathbf{B}_1 = \nabla \times \mathbf{A}_1 = -\mathrm{I}_m \bigsqcup_j \langle \nabla 1|\ \sigma_{1,1}|\ j\rangle\langle j|\ \sigma_{1,1} \times \nabla|\ 1\rangle, \tag{56}$$

where I_m represents the imaginary part. In deriving equation (56), we have used the completeness relation (33) and the following relation

$$\nabla \times (\mu\mathbf{b}) = \nabla\mu \times \mathbf{b} + \mu\nabla \times \mathbf{b}, \tag{57}$$

valid for arbitrary scalar μ and vector $\mathbf{b}$.

According to equation (49), $\mathbf{B}_1$ in equation (56) is well defined provided $E_1 \neq E_2$, such that the monopole is expected to be absent in this case. To rigorously establish this, let us calculate the divergence of the Berry curvature, i.e. $\nabla \cdot \mathbf{B}_1$. Introducing an auxiliary operator

$$\mathbf{F} = -\mathrm{i}\sigma_{1,1}\bigsqcup_j |\nabla j\rangle\langle j|\sigma_{1,1}, \tag{58}$$

which is Hermitian, $\mathbf{F} = \mathbf{F}^\dagger$, as ensured by the completeness relation (33), we have

$$\begin{aligned} \sigma_{1,1}|\nabla j\rangle &= \mathrm{i}\mathbf{F}|j\rangle \\ \nabla \times \mathbf{F} &= -\mathrm{i}\sigma_{1,1}\bigsqcup_j |\nabla j\rangle \times \langle \nabla j|\sigma_{1,1} \\ &= -\mathrm{i}\bigsqcup_j \mathbf{F}|j\rangle \times \langle j|\mathbf{F} \\ &= -\mathrm{i}\mathbf{F} \times \sigma_{1,1}\mathbf{F}. \end{aligned} \tag{59}$$

In deriving the above, we have used equation (57). Further noting that

$$\mathrm{i}\langle j|\mathbf{F}|k\rangle = \bigsqcup_{j'} \langle j|\sigma_{1,1}|\nabla j'\rangle\langle j'|\sigma_{1,1}|k\rangle = \langle j|\sigma_{1,1}\nabla|k\rangle, \tag{60}$$

the Berry curvature can be expressed in terms of $\mathbf{F}$ as

$$\begin{aligned} \mathbf{B}_1 &= -\mathrm{I}_m\bigsqcup_j \langle 1|\mathbf{F}|j\rangle \times \langle j|\mathbf{F}|1\rangle \\ &= -\mathrm{I}_m\langle 1|\mathbf{F} \times \sigma_{1,1}\mathbf{F}|1\rangle. \end{aligned} \tag{61}$$

Finally, by virtue of $\nabla \times \mathbf{F}$ in equation (59), we find

$$\begin{aligned} \nabla \cdot \mathbf{B}_1 = &-\mathrm{I}_m[\langle \nabla 1|\cdot(\mathbf{F} \times \sigma_{1,1}\mathbf{F})|\ 1\rangle + \langle 1|(\mathbf{F} \times \sigma_{1,1}\mathbf{F}) \cdot \nabla|\ 1\rangle \\ &+ \langle 1|\ \nabla \cdot (\mathbf{F} \times \sigma_{1,1}\mathbf{F})|\ 1\rangle] \\ = &-\mathrm{I}_m[-\mathrm{i}\langle 1|\ \mathbf{F}\sigma_{1,1} \cdot (\mathbf{F} \times \sigma_{1,1}\mathbf{F})|\ 1\rangle + \mathrm{i}\langle 1|(\mathbf{F} \times \sigma_{1,1}\mathbf{F}) \cdot \sigma_{1,1}\mathbf{F}|\ 1\rangle \\ &+ \langle 1|(\nabla \times \mathbf{F}) \cdot \sigma_{1,1}\mathbf{F}|\ 1\rangle - \langle 1|\mathbf{F}\sigma_{1,1}\cdot(\nabla \times \mathbf{F})|\ 1\rangle] \\ = &\ 0. \end{aligned} \tag{62}$$

Therefore, as expected, the monopole in the Lorentz quantum mechanics can only appear in the degenerate regime where $\mathbf{B}_1$ diverges, similar as the conventional unitary quantum mechanics.

Next, searching for the monopole, we focus on the degeneracy regime in the parameter space defined by (m_1, m_2, m_3), which, as shown in figure 1, forms a circular cone. There, imagine the path of $\mathbf{R} = (m_1, m_2, m_3)$ realizes a loop in the vicinity of the cone's surface. In this case, the instantaneous eigenstate, say, $|1(\mathbf{R})\rangle$, is

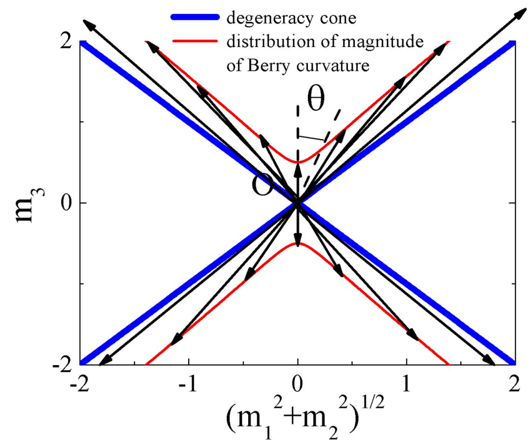

Figure 3. Illustration of the analytic result given by equation (63) for the distribution of strength of Berry curvature (magnetic field) for instantaneous eigenstate $|2\rangle$. For the state $|1\rangle$, everything is the same except that the direction of Berry curvature is reversed, which we drop for clarity. The magnetic fluxes are always straight lines which emanate from the origin O (the tip of the cone) in $(m_1,\ m_2,\ m_3)$ space as parameterized in equation (16). θ introduced in equation (63) is the angle spanned by m_3 axis and direction of Berry curvature under study. There is no magnetic flux outside of the cone, in the cone the magnetic field becomes stronger as approaching the cone's surface and tends to infinity on the surface. Because the flux density assumes the axial symmetry about the m_3 axis, the two dimensional plot is depicted for clarity.

expected to vary in a back-and-forth manner (dropping the overall phases including both the dynamical and Berry phase). This is because the instantaneous eigenstate, apart from an overall phase, is always the same along any straight line emanating from the origin. As a result, the integration of $\mathbf{A}_1$ along this loop vanishes, meaning there is no charge of the Berry curvature on the cone's surface, even though it is in the degeneracy regime.

We thus conclude that—just as in the case of unitary spinor—the charge, if exists, can only be distributed on the isolated points, i.e., the original monopole O, in $\mathbf{R} = (m_1,\ m_2,\ m_3)$ space. However, different from unitary spinor, the magnetic flux does not uniformly emanate from the monopole O to the parameter space, instead, it emanates only to the region in the cone (more closer to the m_3 axis). In addition, even in this region, the magnetic flux is not uniformly distributed. Specifically, by evaluating the geometric phase along a loop perpendicular to the m_3 axis, we can find the distribution of the magnetic flux density per *solid angle* as a function of the angle θ from m_3 axis, i.e.,

$$\rho = \mp\frac{(1 + \tan^2\theta)^{\frac{3}{2}}}{2(1 - \tan^2\theta)^{\frac{3}{2}}}, \tag{63}$$

with $-/+$ associated with the state $|1\rangle$ ($|2\rangle$). Note that the flux density is proportional to the Berry curvature, which acts as a magnetic field, whose magnitude according to equation (63) increases when approaching the cone. Right on the surface of the cone, where $\theta \to \frac{\pi}{4}$, the magnetic field diverges. Outside the cone, on the other hand, the eigenvalue becomes complex such that the notion of adiabatic evolution and geometric phase become meaningless, i.e. there is no magnetic field emanating outside the cone from the monopole O. Again, due to the aforementioned fact that the instantaneous eigenstate (apart from an overall phase) remains the same along any straight line emanating from the origin, we expect all the magnetic field fluxes to be described by straight lines (see figure 3).

Alternatively, we can write $\sigma_{1,1}H$ in terms of the analogues of Pauli's matrices τ_i (see equation (36)), which is then mapped onto a vector $(n_1,\ n_2,\ n_3)$ in the parameter space. However, this equivalent kind of decomposition will not contribute anything but modify the slope of Berry curvature $\theta \to \theta'$ ($\tan(\theta) = 1/C$, while $\tan(\theta') = 1/(C - \sqrt{2})$, with C being any constant).

3.3. Chern number

The Chern number—which reflects the total magnetic charge contained by the monopole on O—can be calculated from equation (63) as,

$$\mathcal{C}_n = \mp\infty, \tag{64}$$

with $-/+$ for the state $|1\rangle$ ($|2\rangle$). Hence, the Lorentz spinor not only has distinct distribution of the magnetic flux compared to the unitary spinor, both also possesses unexpectedly the qualitatively different Chern number which is divergent.

4. Physical examples

In previous sections, we have developed and studied the Lorentz quantum mechanics for the simplest Lorentz spinor. Such a Lorentz spinor can arise in physical systems containing bosonic Bogoliubov quasiparticles, for example, in BECs [5]. Specifically, we illustrate our study of Lorentz quantum mechanics by investigating a 1D fermion gas at low temperatures, phonon excitations on top of a vortex in the BEC, and spin wave excitations in a 1D antiferromagnetic system.

4.1. One-dimensional Fermi gas

As the first illustrative example, we investigate the fermion excitations in a 1D fermion gas at low temperatures. Since excitations dominantly occur for fermions near the Fermi surface (note at 1D, the Fermi surface shrinks to the left (L) and right (R) Fermi points), the corresponding Hamiltonian can then be written as [2]

$$H_{\rm F} = \sum_{s=R,L}\sum_{q}(a_{sq}^{\dagger}v_F q a_{sq}\kappa_s + \frac{1}{2N}g_4\rho_{sq}\rho_{s-q} + g_2\rho_{sq}\rho_{\bar{s}-q}). \tag{65}$$

Here, the operator $a_{sq}^{\dagger}$ (a_{sq}) creates (annihilates) an excited fermion near the Fermi point ($s = {\rm R},\ {\rm L}$) with momentum q (measured with respect to the ground state value). In addition, $\kappa_s = 1,\ -1$ for $s = R/L$, $\bar{s} = L/R$, v_F labels the Fermi velocity, and $\rho_{sq} = \sum_k a_{sk+q}^{\dagger}a_{sk}$ is the density operator in the momentum space representation. In writing down equation (65), we have taken into account the interactions between two fermions. Specifically, g_2 denotes the strength of interaction between two fermions near opposite Fermi points (i.e. $q \simeq 2k_F$), while g_4 for those close to the same Fermi point (i.e. $q \simeq 0$).

Let $|0\rangle$ denote the state of perfect Fermi sphere (a Fermi line in 1D case). A generic state describing density fluctuations near the Fermi points can then be written in terms of a pseduo-spinor as

$$\begin{pmatrix} a \\ b \end{pmatrix} \equiv \frac{1}{\rho}\left(a\sqrt{\frac{2\pi}{lq}}\rho_{Lq} + b\sqrt{\frac{2\pi}{lq}}\rho_{Rq}\right)|0\rangle, \tag{66}$$

where l is the size of the system. As discussed in [2], the density operators ρ_{sq} can be effectively treated as bosonic operators within the approximation

$$[\rho_{sq},\ \rho_{s'q'}] \simeq \langle 0[\rho_{sq},\ \rho_{s'q'}]|0\rangle. \tag{67}$$

By assuming equation (67), it is found that equation (66) represents a Lorentz spinor whose dynamics is governed by the BdG equation below

$$i\hbar\frac{\rm d}{{\rm d}t}\begin{pmatrix} a \\ b \end{pmatrix} = \sigma_{1,1}q\begin{pmatrix} v_F + \frac{g_4}{2\pi} & \frac{g_2}{2\pi} \\ \frac{g_2}{2\pi} & v_F + \frac{g_4}{2\pi} \end{pmatrix}\begin{pmatrix} a \\ b \end{pmatrix}. \tag{68}$$

The generator $\sigma_{1,1}H$ of the dynamics in equation (68), when written in form of equation (16), corresponds to $m_1 = g_2 q/(2\pi)$, $m_2 = 0$ and $m_3 = v_F q + g_4/(2\pi)q$. Thus, when $v_F + \frac{g_4}{2\pi} \geqslant \frac{g_2}{2\pi}$ (see equation (18)), the $\sigma_{1,1}H$ exhibits real eigenvalues, and has space-like and a time-like eigenvectors. Due to $m_2 = 0$, as illustrated in figure 3, there is no magnetic flux penetrating a loop in the plane defined by $(m_1,\ m_3)$. As a result, the Berry phase picked up by the eigenstate, say $|1(\mathbf{R})\rangle$, is always zero when $\mathbf{R}$ varies along a loop in the parameter space of $(m_1,\ m_3)$. According to our theory, it is impossible to implement a geometric force (vector potential or artificial magnetic field) to any fermions in the 1D Fermi gas. We must search for other intriguing systems to implement an artificial magnetic field. Below is an example.

4.2. Phonon excitations on top of a Bose–Einstein condensate vortex

The above example shows that the existence of a non-zero Berry phase requires $\sigma_{1,1}H$—when written in form of (16)—to contain a complex part, i.e. $m_2 \neq 0$. Below, we demonstrate that this can be realized in the dynamics of phonons excited on top of a vortex in a BEC.

Following [6], we assume the phonon wave packet has a narrow width smaller than all the relevant length scales associated with slowly varying potentials (e.g. trapping potential). The corresponding effective BdG equation can be derived as

$$i\hbar\frac{\rm d}{{\rm d}t}\begin{pmatrix} a \\ b \end{pmatrix} = \sigma_{1,1}\begin{pmatrix} H_+ & H_2{\rm e}^{2i\alpha(\mathbf{r})} \\ H_2{\rm e}^{-2i\alpha(\mathbf{r})} & H_- \end{pmatrix}\begin{pmatrix} a \\ b \end{pmatrix}, \tag{69}$$

where $H_2 = gn(\mathbf{r}_c)$ and

$$H_{\pm} = \frac{\mathbf{q}^2}{2} + 2gn(\mathbf{r}_c) + V(\mathbf{r}_c) - \mu \mp \boldsymbol{\Omega} \cdot (\mathbf{r}_c \times \mathbf{q}). \tag{70}$$

Here, $\mathbf{r}_c$ labels the coordinate of the vortex center, g is the interatomic coupling constant, $V(\mathbf{r}_c)$ is the trapping potential of BEC, and $\boldsymbol{\Omega}$ is the rotating frequency of the whole system. Furthermore, $n(\mathbf{r}_c)$ and $\alpha(\mathbf{r}_c)$ denote the particle density and phase of the wavefunction around the vortex center, respectively, with $\mathbf{q}$ labeling the wave vector of phonons.

For every value of $(\mathbf{q}, \mathbf{r}_c)$, the $\sigma_{1,1}H$ read off from equation (69) can be cast into the form (16) with

$$\begin{aligned} m_1 &= gn(\mathbf{r}_c)\cos[2\alpha(\mathbf{r}_c)], \\ m_2 &= gn(\mathbf{r}_c)\sin[2\alpha(\mathbf{r}_c)], \\ m_3 &= \mathbf{q}^2/2 \ + \ 2gn(\mathbf{r}_c) + V(\mathbf{r}_c) - \mu. \end{aligned} \tag{71}$$

In this case, the space-like eigenstate of $\sigma_{1,1}H$ reads

$$|1\rangle = \frac{1}{2}\begin{pmatrix} \zeta + \zeta^{-1} \\ (\zeta - \zeta^{-1})\mathrm{e}^{-2\mathrm{i}\alpha(\mathbf{r}_c)} \end{pmatrix}, \tag{72}$$

with $\zeta = \left(\frac{H_1 - m_3}{H_1 + m_3}\right)^{1/4}$. The eigenstate (72) features a complex angle. As a result, when $\mathbf{r}_c$ varies in the real space, the eigenstate $|1\rangle$ will pick up a non-zero Berry phase; calculating the Berry connection

$$\mathbf{A}_1 = \mathrm{i}\langle 1|\sigma_{1,1}\frac{\partial}{\partial \mathbf{r}_c}|1\rangle,$$

we derive the Berry phase as

$$\beta_1 = \oint \mathrm{d}\mathbf{r}_c \cdot \mathbf{A}_1 = -\oint (M - 1)\mathrm{d}\alpha(\mathbf{r}_c), \tag{73}$$

with M the total atomic mass contained in the quasiparticle wave packet. The Berry connection $\mathbf{A}_1$ will then give rise to an effective vector potential (magnetic field) acting on the spatial motion of the vortex. In a previous study of the system [6], the vector potential has been worked out for a regime of the parameter space but the global feature of the distribution of the Berry-like curvature (magnetic field) is still left unknown. In our calculation, the distribution of magnetic field for the two-mode BdG equation is globally depicted in figure 3.

4.3. Spin-wave excitations in antiferromagnet

Here we demonstrate the Lorentz spin-orbital coupling (SOC) for the spin-wave excitations in a 1D antiferromagnet. Concretely, we consider two sublattices, labeled by A and B, which encode the positive and negative magnetic moments near zero temperature. The corresponding Hamiltonian in the standard Heisenberg's description reads

$$\begin{aligned} H_s = J\sum_{i,\delta}[S_{ai}^z S_{b,i+\delta}^z + \frac{1}{2}(S_{ai}^+ S_{b,i+\delta}^- + S_{ai}^- S_{b,i+\delta}^+)] \\ + J\sum_{j,\delta}[S_{bj}^z S_{a,j+\delta}^z + \frac{1}{2}(S_{bj}^+ S_{a,j+\delta}^- + S_{bj}^- S_{a,j+\delta}^+)], \end{aligned} \tag{74}$$

where $\delta = \pm 1$ stands for the nearest neighboring sites, $J > 0$ is the antiferromagnetic exchange integral, S_{ai}^z (S_{bj}^z) are the spin operator (z component) on the sublattice A (B), and $S^{\pm}$ is the standard spin flip operators. Without loss of generality, we suppose the spins in the sublattice A (B) are along the positive (negative) z direction in the limit of low temperatures.

Hamiltonian (74) can be recast into a more transparent form using the Holstein–Primakoff transformation [27]. Briefly, introducing $a_i^\dagger = S_{ai}^-$, and $b_i^\dagger = S_{bi}^+$, together with the Fourier transformation into the momentum space

$$a_i = N^{-\frac{1}{2}}\sum_k \mathrm{e}^{\mathrm{i}kR_i}a_k, \quad a_i^\dagger = N^{-\frac{1}{2}}\sum_k \mathrm{e}^{-\mathrm{i}kR_i}a_k^\dagger, \tag{75}$$

$$b_j = N^{-\frac{1}{2}}\sum_k \mathrm{e}^{-\mathrm{i}kR_j}b_k, \quad b_j^\dagger = N^{-\frac{1}{2}}\sum_k \mathrm{e}^{\mathrm{i}kR_j}b_k^\dagger, \tag{76}$$

we rewrite equation (74) as (dropping a constant)

$$\begin{aligned}\tilde{H}_s &= 2ZSJ\sum_k(a_k^\dagger a_k + b_k^\dagger b_k + \gamma_k a_k^\dagger b_k^\dagger + \gamma_k b_k a_k)\\ &= 2ZSJ\sum_k\begin{pmatrix}a_k^\dagger & b_k\end{pmatrix}\begin{pmatrix}1 & \gamma_k\\ \gamma_k & 1\end{pmatrix}\begin{pmatrix}a_k\\ b_k^\dagger\end{pmatrix}.\end{aligned} \tag{77}$$

Here, $Z = 2$ is the coordination number for the 1D system; $\gamma_k = \frac{1}{Z}\sum_\delta \mathrm{e}^{\mathbf{i}\mathbf{k}\cdot\boldsymbol{\delta}} = \cos(k)$ is the structure factor of the 1D lattice (here the lattice constant is taken as $a_l = 1$, and the momentum is measured in the unit of $\hbar/a_l$). Let the ground state of Hamiltonian (77) be denoted as $|0\rangle$ (which involves a superposition of enormous number of Fock states in the particle number representation $a_k^\dagger a_k$, $b_k^\dagger b_k$).

The above Holstein–Primakoff transformation allows a vivid description of the spin-wave excitations of the system (see equation (74)) in terms of 'particles' and 'holes' created in the ground state. In the simplest case, we consider the dynamics of an arbitrary (1, 1)-spinor state given by

$$\begin{pmatrix}a\\ b\end{pmatrix} \equiv \frac{1}{\rho}(aa_k^\dagger + bb_k)|0\rangle, \tag{78}$$

with ρ the normalization constant, corresponding to creations of a pair of particle and hole. The time evolution of equation (78) can be derived as

$$\mathrm{i}\hbar\frac{\mathrm{d}}{\mathrm{d}t}\begin{pmatrix}a\\ b\end{pmatrix} = \sigma_{1,1}\begin{pmatrix}1 & \gamma_k\\ \gamma_k & 1\end{pmatrix}\begin{pmatrix}a\\ b\end{pmatrix}, \tag{79}$$

which features a k-dependent generator. The corresponding eigenspinors $(u,\ v)^T$ and $(v^*,\ u^*)^T$ are found to be real and take the form

$$u(k) = \sqrt{\frac{1}{2}\left(\frac{1}{|\sin(k)|} + 1\right)}, \tag{80}$$

$$v(k) = \mathrm{sgn}(\cos(k))\sqrt{\frac{1}{2}\left(\frac{1}{|\sin(k)|} - 1\right)}, \tag{81}$$

which manifestly exhibit the SOC effect, with the orbital state k coupled to a Lorentz spinor. Since the SOC effect for the conventional unitary quantum mechanics has been studied extensively in both single-body systems [28–31], where Zitterbewegung oscillation occurs [28, 29] and BEC systems [32], where single plane wave phase and standing wave phase were found, along this direction we may expect and explore the ample physical consequences of the Lorentz SOC.

5. Conclusion

To summarize, we have studied the dynamics of bosonic quasiparticles based on the BdG equation for the (1, 1)-spinor. We show that the dynamical behavior of these bosonic quasiparticles is described by Lorentz quantum mechanics, where both time evolution of a quantum state and the representation transformation represent Lorentz transformations in the complex Minkowski space. The basic framework of the Lorentz quantum mechanics for the Lorentz spinor is presented, including construction of basic operators that are analogue of Pauli matrices. Based on this, we have demonstrated the Lorentz counterpart of the Berry phase, Berry connection, and Berry curvatures, etc. Since such Lorentz spinors can be generically found in physical systems hosting bosonic Bogoliubov quasi-particles, we expect that our study allows new insights into the dynamical properties of quasiparticles in diverse systems. In a broader context, the present work provides a new perspective towards the fundamental understanding of quantum evolution, as well as new scenarios for experimentally probing the coherent effect. While our study is primarily based on bosonic Bogoliubov equation for the (1, 1)-spinor, we expect the essential features also appear in dynamics described by the bosonic Bogoliubov equation of multi-mode, the study of which is of future interest.

References

[1] Bogoliubov N N 1947 *J. Phys. USSR.* **11** 23
[2] Giamarchi T 2004 *Quantum Physics in One Dimension* (Oxford: Oxford University Press)
[3] Pethick C J and Smith H 2002 *Bose–Einstein Condensation in Dilute Gases* (Cambridge: Cambridge University Press)
[4] Volovik G E 2003 *The Universe in a Helium Droplet* (Oxford: Clarendon Press)
[5] Wu B and Niu Q 2003 *New J. of Phys.* **5** 104
[6] Zhang C, Dudarev A M and Niu Q 2006 *Phys. Rev. Lett.* **97** 040401

[7] Bender C M and Boettcher S 1998 *Phys. Rev. Lett.* **80** 5243
Bender C M, Boettcher S and Meisinger P N 1999 *J. Math. Phys.* **40** 2201
[8] Zhong-Peng L *et al* 2016 *Phys. Rev. Lett.* **117** 110802
[9] Jing H *et al* 2014 *Phys. Rev. Lett.* **113** 053604
[10] Lü X-Y, Jing H, Ma J-Y and Wu Y 2015 *Phys. Rev. Lett.* **114** 253601
[11] Jing H *et al* 2015 *Sci. Rep.* **5** 9663
[12] Zhang J *et al* 2015 *Phys. Rev.* B **92** 115407
[13] Jiangbin G and Qing-hai W 2010 *Phys. Rev.* A **82** 012103
Jiangbin G and Qing-hai W 2013 *J. Phys. A: Math. Theor.* **46** 485302
[14] Jiangbin G and Qing-hai W 2015 *Phys. Rev.* A **91** 042135
[15] Qing-hai W, Song-zhi C and Jie-hong Z 2010 *J. Phys. A: Math. Theor.* **43** 295301
[16] Peng P *et al* 2016 *Nat. Phys.* **12** 1139
[17] EL-Ganainy R, Makris K G, Christodoulides D N and Musslimani Z H 2007 *Opt. Lett.* **32** 2632
[18] Bender C M, Brody D C and Jones H F 2002 *Phys. Rev. Lett.* **89** 270401
Bender C M, Brody D C and H F 2004 *Phys. Rev. Lett.* **92** 119902 (erratum)
[19] Bender C M, Meisinger P N and Wang Q 2003 *J. Phys. A: Math. Gen.* **36** 6791
[20] Mostafazadeh A 2003 *J. Phys. A: Math. Gen.* **36** 7081–92
[21] Mostafazadeh A A and Ozcelik S 2006 *Turk. J. Phys.* **30** 437–43
[22] Swanson M S 2004 *J. Math. Phys.* **45** 585
[23] Fring A and Moussa M H Y 2016 *Phys. Rev.* A **94** 042128
[24] Born M and Fock V A 1928 *Z. Phys.* A **51** 165
[25] Zhang Q, Gong J and Wu B 2014 *New J. of Phys.* **16** 123024
[26] Berry M V 1984 *Proc. R. Soc.* A **392** 45
[27] Holstein T and Primakoff H 1940 *Phys. Rev.* **58** 1098
[28] Schrödinger E 1930 *Sitzber. preuss. Akad. Wiss., Physikmath. Kl.* **24** 418
[29] Vaishnav J Y and Clark C W 2008 *Phys. Rev. Lett.* **100** 153002
[30] Ruseckas J, Juzeliunas G, Ohberg P and Fleischhauer M 2005 *Phys. Rev. Lett.* **95** 010404
[31] Juzeliunas G, Ruseckas J, Jacob A, Santos L and Ohberg P 2008 *Phys. Rev. Lett.* **100** 200405
[32] Wang C, Gao C, Jian C-M and Zhai H 2010 Spin-orbit coupled spinor Bose–Einstein condensates *Phys. Rev. Lett.* **105** 160403