Two Critical velocities for a superfluid in a periodic potential

Biao Wu and Junren Shi

Institute of Physics and ICQS, Chinese Academy of Sciences, Beijing 100080, China
(Dated: November 9, 2008)

In contrast to a homogeneous superfluid which has only one critical velocity, there exist two critical velocities for a superfluid in a periodic potential. The first one, which we call inside critical velocity, is for a macroscopic impurity to move frictionlessly in the periodic superfluid system; the second, which is called trawler critical velocity, is the largest velocity of the lattice for the superfluidity to maintain. The result is relevant to the superfluidity observed in the Bose-Einstein condensate in an optical lattice and supersolid helium.

One characteristic feature of superfluidity is the existence of critical velocity. According to Landau's theory of superfluidity[1], this critical velocity is given by the speed of sound. In developing his theory, Landau had his focus on a homogeneous superfluid, which has continued to be the focus of later studies[2, 3]. Experimental advances in recent years have brought to people's attention a different type of systems, superfluids in periodic potentials. They include a Bose-Einstein condensate (BEC) in an optical lattice[4] and supersolid helium[5]. In the BEC system, the lattice is created by counter-propagating laser beams. For supersolid helium, the superfluid is defects (vacancies or interstitials) in the lattice self-assembled by helium atoms as commonly believed[6, 7]. Interestingly, the crust of a neutron star can also be considered as a superfluid in a lattice[8, 9]. In addition, it is expected that the superfluidity of a paired Fermi gas in an optical lattice will be a subject of intensive investigation[10].

In this Letter we show that the presence of the periodic potential has non-trivial consequences, requiring a revisit of the concept of critical velocity. In contrast to the homogeneous superfluid which has only one critical velocity, there are two distinct critical velocities for a superfluid in a periodic potential. The first one, which we call inside critical velocity, is for an impurity to move frictionlessly in the periodic superfluid system (Fig.1(a)); the second, which is called trawler critical velocity, is the largest velocity of the lattice for the superfluidity to maintain (Fig.1(b)). These two critical velocities will be illustrated with a BEC in a one-dimensional optical lattice.

The presence of the periodic potential plays a decisive role in the appearance of the two critical velocities. Because of the addition of a periodic potential, two very different situations can arise in the superfluid system. The first situation is described in Fig.1(a), where one macroscopic impurity moves inside the superfluid. The key feature in this situation is that there is no relative motion between the superfluid and the periodic potential. Fig.1(b) illustrates the other situation, where the lattice is slowly accelerated to a given velocity and there is a relative motion between the superfluid and the periodic

potential. For these two different situations, naturally arise two critical velocities.
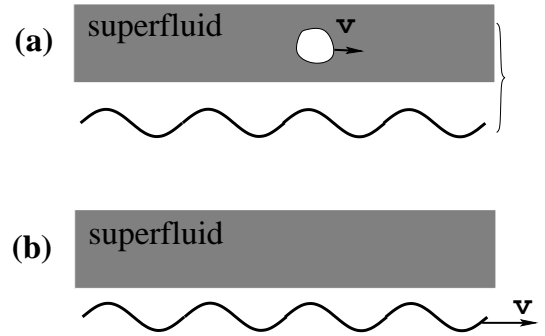

FIG. 1: (a) A macroscopic obstacle moves with a velocity of **v** inside a superfluid residing in a periodic potential. The curly brace indicates that the superfluid and the periodic potential are "locked" together and there is no relative motion between them. (b) The lattice where a superfluid resides is slowly accelerated to a velocity of **v**.

Both critical velocities can be measured with BECs in optical lattices. The inside critical velocity v_i can be measured with the same experimental setting as in Ref.[11], where the superfluidity of a BEC was studied by moving a blue-detuned laser inside the BEC. For the trawler critical velocity v_t, one can repeat the experiment in Ref.[12] where a BEC is loaded in a moving optical lattice. One only needs to shift his attention from dynamical instability to superfluidity. For solid helium, only the trawler critical velocity can be measured.

It is instructive to briefly review Landau's theory of critical velocity for superfluid before further discussion[2]. Consider a superfluid moving inside a small tube with a velocity of **v** as in Fig.2. Suppose a single elementary excitation is generated and its energy is $\varepsilon_0(\mathbf{p})$ and momentum is **p** in the superfluid frame where the superfluid is at rest, then by the usual Galilean transformation, we have the excitation energy:

$$\varepsilon(\mathbf{v}, \mathbf{p}) = \varepsilon_0(\mathbf{p}) - \mathbf{v} \cdot \mathbf{p}, \tag{1}$$

in the reference frame where the tube is motionless. Landau argues that if $\varepsilon(\mathbf{v}, \mathbf{p})$ is negative for some excitations,

these excitations are energetically favored to be generated and, therefore, the superfluidity is unstable. Since the low energy excitations are phonons $\varepsilon_0(\mathbf{p}) = up$, it is clear that $\varepsilon(\mathbf{v}, \mathbf{p})$ can be negative only when $v > u$. This implies that the sound speed u is the critical velocity beyond which the superfluid becomes normal fluid.

FIG. 2: A superfluid moves inside a small tube with a velocity of $\mathbf{v}$.

It is clear that the "tube" frame holds a special position in Landau's theory. It is only in this frame the tube does not disturb the system so that the normal fluid consisting of the excitation "gases" can be in a thermal equilibrium with the environment [13]. However, in the presence of the periodic potential, the disturbance can be from either the external impurity (Fig. 1(a)) or the imperfection of the periodic potential (Fig. 1(b)), and one needs to define the different "tube" frames for the different scenarios. For this reason and convenience of our future discussion, we call such a frame as thermodynamics frame.

We focus first on the situation of Fig.1(a), where a macroscopic obstacle moves inside the periodic superfluid with a velocity $\mathbf{v}$. We follow closely Landau's argument[2] and start our discussion in the superfluid frame where the superfluid is at rest. In such a periodic system, the elementary excitation (quasi-particle) is characterized by quasi-momentum $\mathbf{q}$ and the Bloch-band index n. If the excited state is described by the wavefunction $\Psi_{n\mathbf{q}}$, then its energy and momentum are, respectively,

$$\varepsilon_n(\mathbf{q}) = \langle \Psi_{n\mathbf{q}} | \hat{H} | \Psi_{n\mathbf{q}} \rangle - E_0 \,, \qquad (2)$$
$$\mathbf{p}_n(\mathbf{q}) = \langle \Psi_{n\mathbf{q}} | \hat{\mathbf{p}} | \Psi_{n\mathbf{q}} \rangle \,, \qquad (3)$$

where $\hat{H}$ is the Hamiltonian of system with E_0 being the ground state energy. $\hat{\mathbf{p}} = \sum_j \mathbf{p}_j$ is the total momentum operator of the system. Note that without the periodic potential one would have simply $\mathbf{p} = \hbar \mathbf{q}$. With the periodic potential, this simple relation no longer holds. In the thermodynamic frame in which the obstacle is motionless, the Hamiltonian of the system is transformed to:

$$\hat{H}' = \hat{H} - \mathbf{v} \cdot \hat{\mathbf{p}} + \frac{1}{2} M \mathbf{v}^2 \qquad (4)$$

where M is the total mass of the system. The excitation energy in this frame reads,

$$\varepsilon_n'(\mathbf{v}, \mathbf{q}) = \langle \Psi_{n\mathbf{q}} | \hat{H}' | \Psi_{n\mathbf{q}} \rangle - E_0' = \varepsilon_n(\mathbf{q}) - \mathbf{v} \cdot \mathbf{p}_n(\mathbf{q}) \,, \quad (5)$$

where $E_0' = E_0 + Mv^2/2$ is the ground state energy in the thermodynamics frame. This excitation energy determines the stability of the system. If it is positive for

all values of $\mathbf{q}$ and band index n, then the excitation of quasi-particles is not energetically favored and the system is a superfluid. Otherwise, the quasi-particles can be generated spontaneously and the liquid flow experiences viscosity. We can thus define a critical velocity, beyond which $\varepsilon_n'(\mathbf{q})$ can be negative for some values of n, $\mathbf{q}$. This critical velocity v_i is given by

$$v_i = \text{Minimum of } \frac{\varepsilon_n(\mathbf{q})}{|\mathbf{p}_n(\mathbf{q})|} \,. \qquad (6)$$

We call v_i inside critical velocity to distinguish it from the other critical velocity that we shall discuss next.
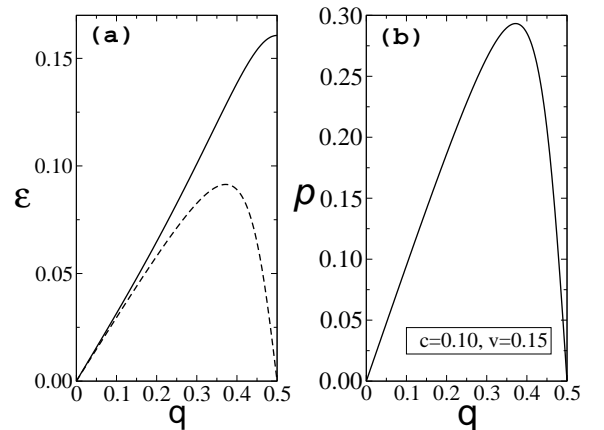

FIG. 3: (a) Energy $\varepsilon_0(q)$ and (b) momentum $p(q)$ of a quasi-particle for a BEC in a one-dimensional optical lattice. The dashed line in (a) is for $up(q)$ with u being the sound speed.

We take a look at an example, a BEC in a one-dimensional optical lattice. We treat such a system with the standard Gross-Pitavskii equation,

$$i \frac{\partial}{\partial t} \psi = -\frac{1}{2} \frac{\partial^2}{\partial x^2} \psi + v(\cos x)\psi + c|\psi|^2 \psi \,. \qquad (7)$$

In the above equation, the energy is in units of $4\pi^2 \hbar^2 m/a^2$, the unit of the momentum is $2\pi \hbar/a$, and q is measured in units of π/a (a is the period of the optical lattice). For the details of the unit system, please refer to Ref.[14]. Following the procedure in Ref.[14], we can compute its Bogoliubov excitations and, consequently, the energies and momenta of these excitations. Plotted in Fig.3 are the energy and momentum of the quasi-particle in this simple system. We see from this figure that the dashed line for $up(q)$ lies completely under $\varepsilon(q)$. This means that the critical velocity v_i is exactly the sound speed u in this simple example.

The motion of an impurity inside a superfluid has been studied for a long time by moving ions inside liquid helium[15]. This technique was used to verify Landau's original criterion of superfluidity[16], which is impossible to verify by flowing superfluid helium inside a small tube as one would generate vortex and turbulence and destroy

superfluidity well before the superflow reaches the sound speed[3]. More recently, an obstacle created by a blue-detuned laser beam was moved back and forth inside a rather homogeneous BEC to test its superfluidity[11]. We believe that this technique can also be used to measure the inside critical velocity for a BEC in an optical lattice. On the theoretical side, the motion of an impurity in a homogeneous superfluid was studied by Girardeau, who found the "onset of acoustical wave drag as the impurity speed reaches the speed of sound"[17].

We study now the case depicted in Fig.1(b), where the lattice is accelerated slowly to a given velocity $\mathbf{v}$. In this case, the thermodynamics frame is the frame that moves along with the lattice. Viewing from the thermodynamics frame, the slow acceleration induces an adiabatic evolution of the Bose-Einstein condensation from the Bloch state at Γ point ($\mathbf{k} = 0$) to a Bloch state with non-vanishing Bloch wave vector $\mathbf{k} = -m\mathbf{v}/\hbar$ [18, 19]. The same effect can be achieved by slowly turning on a moving lattice[12]. The excitation energy in the thermodynamics frame can be obtained in a similar way as that in Eq. (2), albeit the "ground state" for the present case has a macroscopic condensation at nonzero $\mathbf{k}$. We thus have,

$$\tilde{\varepsilon}'_n(\mathbf{v}, \mathbf{q}) = \langle \Psi_{n\mathbf{q};\mathbf{k}} | \hat{H} | \Psi_{n\mathbf{q};\mathbf{k}} \rangle - \tilde{E}_0(\mathbf{k}), \qquad (8)$$

where $\tilde{E}_0(\mathbf{k})$ is the energy of the ground state and $|\Psi_{n\mathbf{q};\mathbf{k}}\rangle$ denotes the excitation state with quasi-momentum $\mathbf{q}$. We note that in the present case both the ground state and the excitation state have dependence on the condensation momentum $\mathbf{k} \equiv -m\mathbf{v}/\hbar$. As a result, the excitation energy $\tilde{\varepsilon}'_n(\mathbf{v}, \mathbf{q})$ depends on both the velocity $\mathbf{v}$ of the periodic potential and the quasi-momentum $\mathbf{q}$ of the excitation. The stability of the superfluid phase can be determined in the same spirit as that in the first case: if $\tilde{\varepsilon}'_n(\mathbf{v}, \mathbf{q})$ can be negative for some finite $\mathbf{q}$, then the superfluidity is lost. The critical velocity for this situation is given by the smallest v such that $\tilde{\varepsilon}'_n(\mathbf{v}, \mathbf{q}) = 0$ for some finite $\mathbf{q}$. We denote it as v_t and call it trawler critical velocity. This trawler critical velocity is related to the Landau instability discussed with the mean-field Gross-Pitaevskii equation in Ref.[20, 21, 22]. Here in this Letter, this Landau instability is discussed in a more general setting.

It is easy to demonstrate that the trawler critical velocity is different from the inside critical velocity. We consider a limiting case that the critical velocity is determined by the low energy phonon excitation (which means that v_t is necessary to be small). In this case, we can limit our consideration in the lowest Bloch band, and the full Hamiltonian of the system can be mapped to an effective one in which the boson moves in the free space (without the periodic potential) but with the renormalized dispersion $E(\mathbf{p})$ of the lowest Bloch band. For $\mathbf{p} \approx \mathbf{0}$, $E(\mathbf{p}) \approx \mathbf{p}^2/2m^*$ with m^* being the effective mass.

The effective Hamiltonian in the thermodynamics frame has the form

$$\hat{H}' = \sum_j \frac{(\hat{\mathbf{p}}_j + \hbar\mathbf{k})^2}{2m^*} + \widetilde{H}_{int},$$

$$= \hat{H}_0 - \frac{m}{m^*}\mathbf{v} \cdot \hat{\mathbf{p}} + \frac{Nm^*}{2}v^2. \qquad (9)$$

where $\hat{H}_0$ is the effective Hamiltonian in the superfluid frame and $\widetilde{H}$ is the interaction potential projected in the lowest Bloch band. When an elementary excitation with quasi-momentum $\hbar\mathbf{q}$ is generated, its energy is according to Eq.(9)

$$\tilde{\varepsilon}'(\mathbf{v}, \mathbf{q}) = \varepsilon_0(\mathbf{q}) - \hbar\frac{m}{m^*}\mathbf{v} \cdot \mathbf{q}, \qquad (10)$$

where $\varepsilon_0(\mathbf{q}) \approx u\hbar\mathbf{q}$ is the phonon excitation energy in the superfluid frame. It is clear that superfluidity is lost when $v > m^*u_0/m$. So, the critical velocity is

$$v_t = \frac{m^*}{m}u_0, \qquad (11)$$

which is different from the inside critical velocity v_i. Note that the velocity $m\mathbf{v}/m^*$ in Eq.(10) is the group velocity of the superfluid.
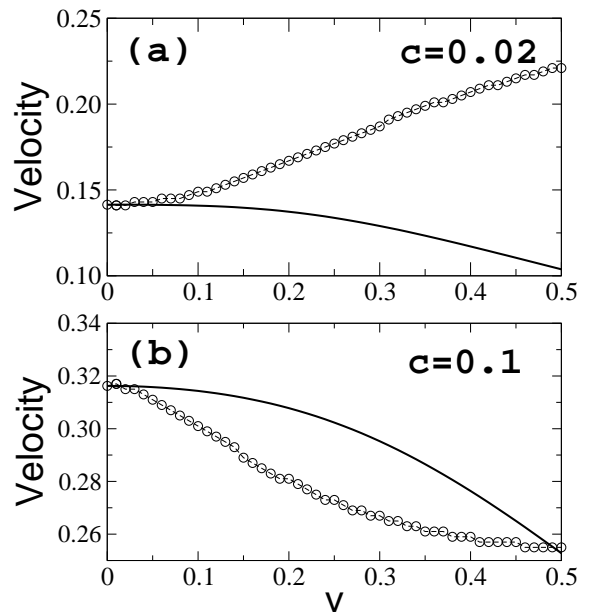

FIG. 4: Inside critical velocity v_i and trawler critical velocity v_t of a BEC in a one dimensional optical lattice as functions of the lattice strength v. (a) $c = 0.02$ (b) $c = 0.1$. The solid line is for v_i and the circles with dashed line represent v_t.

We now demonstrate these two critical velocities, with the simple example, a BEC in a one-dimensional optical lattice. When this system is in the superfluid state, it can be well described by the mean-field Gross-Pitaevskii equation as in Eq.(7). We can find both critical velocities by numerically computing the Bogoliubov spectrum

of this system as in Ref.[14]. We have computed two different cases: in one case the Bloch states corresponding to the trawler critical velocity are close to the Γ point; in the other these Bloch states are far away from the Γ point. For the first case, the results are shown in Fig.4(a), where we see these two velocities have different trends: the inside critical velocities v_i decreases with the lattice strength while the trawler critical velocity increases. In the other case, both critical velocities decrease with the lattice strength.

For supersolid helium, the critical velocity has not been measured up to date. The "critical velocity" (3.6-38μm/s) measured by Kim and Chan[5] is likely associated with the vortex quantization in the annular container. We are aware that the supersolid helium is a highly controversy topic at present[23].

In summary, we have tried to answer a simple question, "What is the critical velocity for a superfluid in a periodic potential?" The answer depends on the way how the critical velocity is probed. If you probe it by moving an impurity inside the BEC, you have the inside critical velocity; if you probe it by moving the lattice, you obtain the trawler critical velocity. These two velocities are different in nature and generally in values. For a BEC in an optical lattice, both critical velocities are measurable with current experimental techniques[11, 12].

[1] L. D. Landau, J. Phys. U.S.S.R. **5**, 71 (1941).

[2] L. D. Landau and E. M. Lifshitz, *Statistical Physics (II)* (Pergamon, Oxford, 1981).

[3] P. Nozieres and D. Pines, *The Theory of Quantum Liquids (II)* (Addison-Wesley, New York, 1990).

[4] O. Morsch and M. Oberthaler, Rev. Mod. Phys. **78**, 179 (2006).

[5] E. Kim and M. H. W. Chan, Science **305**, 1941 (2004).

[6] A. F. Andreev and I. M. Lifshitz, Sov. Phys. JETP **29**, 1107 (1969).

[7] A. J. Leggett, Phys. Rev. Lett. **25**, 1543 (1970).

[8] J. M. Lattimer and M. Prakash, Science **304**, 536 (2004).

[9] B. Carter, N. Chamel, and P. Haensel, Nucl. Phys. A **759**, 441 (2005).

[10] M. W. Zwierlein, J. R. Abo-Shaeer, A. Schirotzek, C. H. Schunck, and W. Ketterle, Nature **435**, 1047 (2005).

[11] R. Onofrio, C. Raman, J. M. Vogels, J. R. Abo-Shaeer, A. P. Chikkatur, and W. Ketterle, Phys. Rev. Lett. **85**, 2228 (2000).

[12] L. Fallani, L. D. Sarlo, J. E. Lye, M. Modugno, R. Saers, C. Fort, and M. Inguscio, Phys. Rev. Lett. **93**, 140406 (2004).

[13] L. P. Pitaevskii, *The Physics of Superconductors (I)* (Springer-Verlag, 2003), chap. 2.

[14] B. Wu and Q. Niu, New J. of Phys. **5**, 104 (2003).

[15] A. L. Fetter, *The Physics of Liquid and Solid Helium (I)* (Wiley, 1976), chap. 3.

[16] F. Reif and L. Meyer, Phys. Rev. **119**, 1164 (1960).

[17] M. Girardeau, Phys. of Fluids **4**, 279 (1961).

[18] D.-I. Choi and Q. Niu, Phys. Rev. Lett. **82**, 2022 (1999).

[19] O. Morsch, J. Müller, M. Cristiani, D. Ciampini, and E. Arimondo, Phys. Rev. Lett. **87**, 140402 (2001).

[20] B. Wu and Q. Niu, Phys. Rev. A **64**, 061603 (2001).

[21] A. Smerzi, A. Trombettoni, P. G. Kevrekidis, and A. R. Bishop, Phys. Rev. Lett. **89**, 170402 (2002).

[22] M. Modugno, C. Tozzo, and F. Dalfovo, Phys. Rev. A **70**, 043625 (2004).

[23] D. Ceperley, Nature Phys. **2**, 659 (2006).