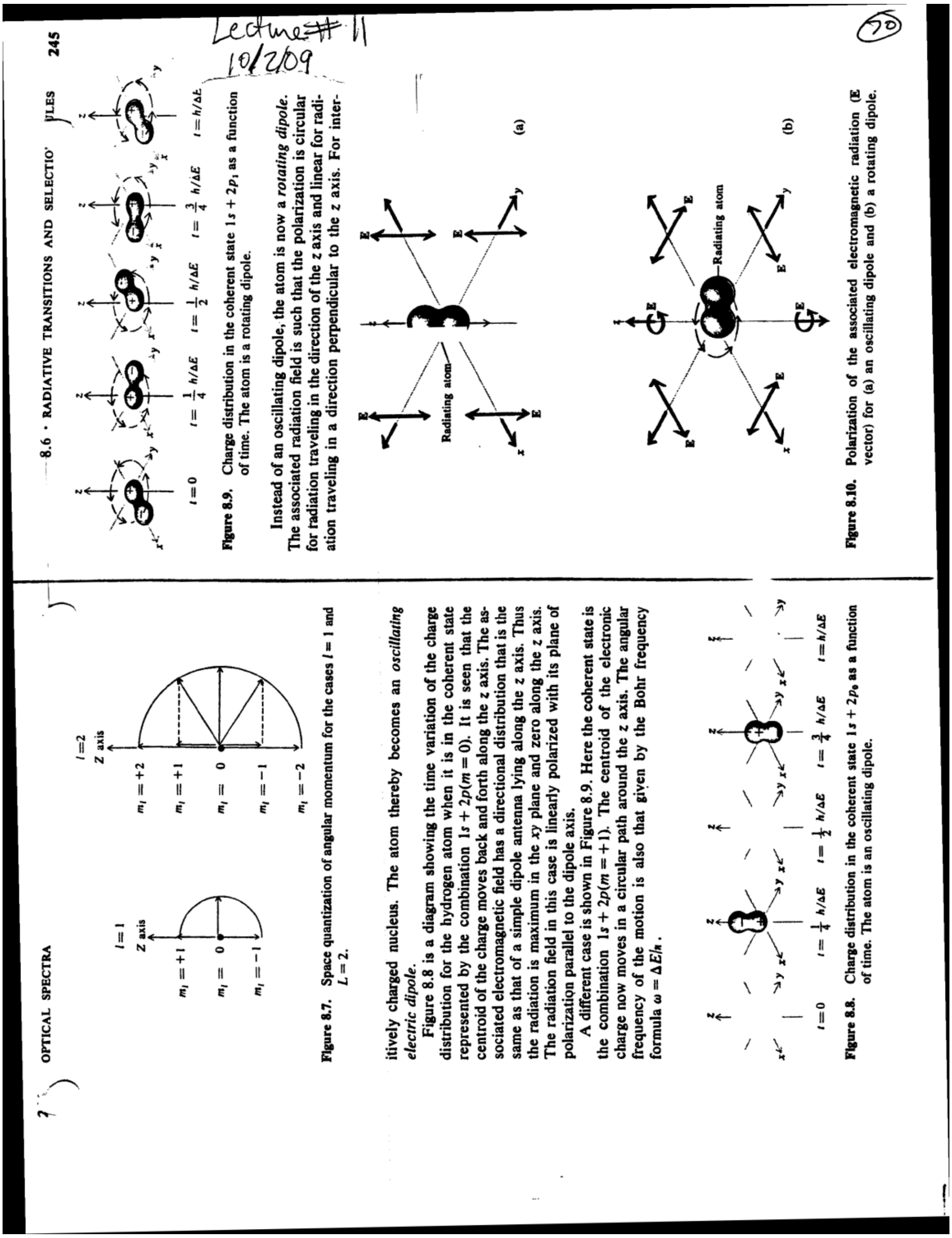
Lecture 11



Lecture # 11
10/2/09

Figure 8.7. Space quantization of angular momentum for the cases $l = 1$ and $L = 2$.

itively charged nucleus. The atom thereby becomes an *oscillating electric dipole.*

Figure 8.8 is a diagram showing the time variation of the charge distribution for the hydrogen atom when it is in the coherent state represented by the combination $1s + 2p(m = 0)$. It is seen that the centroid of the charge moves back and forth along the z axis. The associated electromagnetic field has a directional distribution that is the same as that of a simple dipole antenna lying along the z axis. Thus the radiation is maximum in the xy plane and zero along the z axis. The radiation field in this case is linearly polarized with its plane of polarization parallel to the dipole axis.

A different case is shown in Figure 8.9. Here the coherent state is the combination $1s + 2p(m = +1)$. The centroid of the electronic charge now moves in a circular path around the z axis. The angular frequency of the motion is also that given by the Bohr frequency formula $\omega = \Delta E/\hbar$.

Figure 8.8. Charge distribution in the coherent state $1s + 2p_0$ as a function of time. The atom is an oscillating dipole.

Figure 8.9. Charge distribution in the coherent state $1s + 2p_1$ as a function of time. The atom is a rotating dipole.

Instead of an oscillating dipole, the atom is now a *rotating dipole.* The associated radiation field is such that the polarization is circular for radiation traveling in the direction of the z axis and linear for radiation traveling in a direction perpendicular to the z axis. For inter-

Figure 8.10. Polarization of the associated electromagnetic radiation (E vector) for (a) an oscillating dipole and (b) a rotating dipole.

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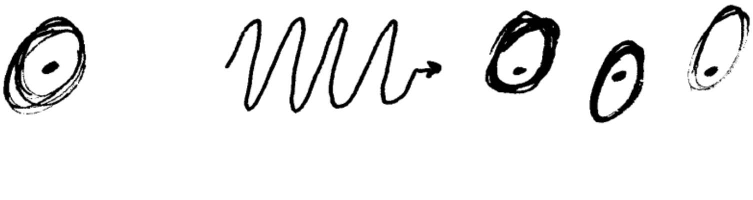
-How is the atom put into such a superposition state?

ε field incident on atom => it "shakes" the electron cloud



How do you describe the displaced wave function in terms of the original states?

$$\psi_{disp} = \sum_{n \neq m} a_n \psi_n = a_{1s}\psi_{1s} + a_{2s}\psi_{2s} + a_{2p}\psi_{2p} + \ldots$$

(Superposition principle)

As usual, find coefficients by

$$a_n = \int \psi_{disp} \psi_n^* dV$$

Note that the 2 largest coefficients will be

$$a_{1s} = \int \psi_{disp} \psi_{1s}^* dV$$

$$a_{2p} = \int \psi_{disp} \psi_{2p}^* dV$$

(if $\hbar\omega_{opt} = E_{2P} - E_{1S}$, then these will be essentially the only nonzero coeffs,) => field puts atoms in superposition state .

Thus, although we cannot make antennas at optical frequencies, we can use the little antennas nature provides us with, namely oscillation atoms and molecules.

Review of electric dipole radiation

In the course of developing the CEO model and its consequences, we are going to need to recall the main features of electric dipole radiation. The proper development of the theory is best left to your electromagnetic course; here we only have time and space to outline the approach and main results.

The wave equation for the vector potential is similar in form to that of the fields:

$$\nabla^2 \vec{A} - \varepsilon_0 \mu_0 \frac{\partial^2 A}{\partial t^2} = -\mu_0 \vec{J}$$

It can be shown that the solution to this equation can be generally expressed as (see, e.g. Jackson dep.6)

$$\vec{A}(\vec{r},t) = \frac{\mu_0}{4\pi} \iint_V \frac{\vec{J}(\vec{r}',t')}{|\vec{r}-\vec{r}'|} dV' dt' \delta(t' + \frac{|\vec{r}-\vec{r}'|}{c} - t)$$

The δ -function guarantees that

$$t' = t - \frac{|\vec{r}-\vec{r}'|}{c} \quad = \text{retarded time}$$

$\Rightarrow \vec{A}$ = “retarded vector potential”

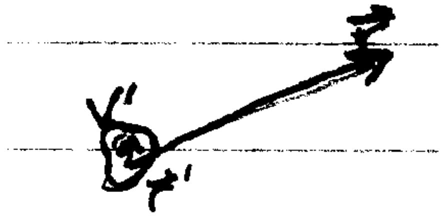
If the current oscillates harmonically

$$\vec{J}(\vec{r},t) = \vec{J}(\vec{r}) e^{i\omega t}$$

Then the delta function gives

$$\vec{A}(\vec{r}) = \frac{\mu_0}{4\pi}\int \vec{J}(\vec{r}')\frac{e^{-ik|\vec{r}-\vec{r}'|}}{|\vec{r}-\vec{r}'|}dV'$$

The region in which $\vec{J} \neq 0$ is very small (atoms, remember), i.e. $\ll \lambda$,so we consider



$$|\vec{r}-\vec{r}'| \simeq r - \hat{r}\cdot\vec{r}' \qquad (\hat{r} = \frac{\vec{r}}{r})$$

In the electric dipole approximation, we keep only the leading term:

$$\vec{A}(\vec{r}) \simeq \frac{\mu_0}{4\pi}\frac{e^{ikr}}{r}\int_{V'}\vec{J}(\vec{r}')dV'$$

(Keeping higher order terms gives radiation from magnetic dipoles, electric quadruples, etc., which need not concern us here.)

Remembering that V' bounds the currents entirely (so $\vec{J}=0$ on the surface of V'), we can put this in a more familiar form by integration by parts :

$$\int \vec{J}(\vec{r}')dV' = -\int \vec{r}'(\nabla\cdot\vec{J})\,\mathrm{d}V' = -i\omega\int \vec{r}'\rho(\vec{r}')\mathrm{d}V'$$

Where we have used $\nabla\cdot\vec{J} = -\frac{\partial A}{\partial t} = -i\omega A$

$$\nabla\cdot J = -\frac{\partial\rho}{\partial t} = -i\omega\rho$$

Define <u>electric dipole moment</u> by

$$\vec{P} = \int_{V'}\vec{r}'\rho(\vec{r}')\mathrm{d}V'$$ (note similarly to quantum expression in P.69)

$$=> \quad \vec{A}(\vec{r}) = -i\omega\frac{\mu_0}{4\pi}\frac{e^{-ikr}}{r}\vec{P}$$

From this we can get the fields via

$$\vec{B} = \nabla\times\vec{A}$$

And $\vec{E} = -i\frac{c^2}{\omega}\nabla\times\vec{B}$ (from $\nabla\times\vec{H} = \frac{\partial D}{\partial t} = i\omega\varepsilon_0\vec{E}$, $\nabla\times\vec{B} = i\omega\mu_0\varepsilon_0\vec{E}$)

We need only consider the special case of dipole oriented along the z-direction:

$$\vec{P} = P_0\hat{z}\cos\omega t$$

Then in the "<u>far field</u>" (r>>l)

$$\vec{E} = (E_r, E_\theta, E_\phi) = (0, \frac{-\omega^2 \rho_0}{4\pi\varepsilon_0 c^2} \sin\theta \frac{\cos\omega(t - \frac{r}{c})}{r}, 0)$$

$$\vec{H} = (H_r, H_\theta, H_\phi) = (0, 0, \frac{-\omega^2 \rho_0}{4\pi c} \sin\theta \frac{\cos\omega(t - \frac{r}{c})}{r})$$

The Poynting vector is thus radial:

$$\vec{S} = (\frac{\omega^4 \rho_0^2}{16\pi\varepsilon_0 c^3} \sin^2\theta \frac{\cos^2\omega(t - \frac{r}{c})}{r^2}, 0, 0)$$

See Figs. on following page

Electromagnetic waves

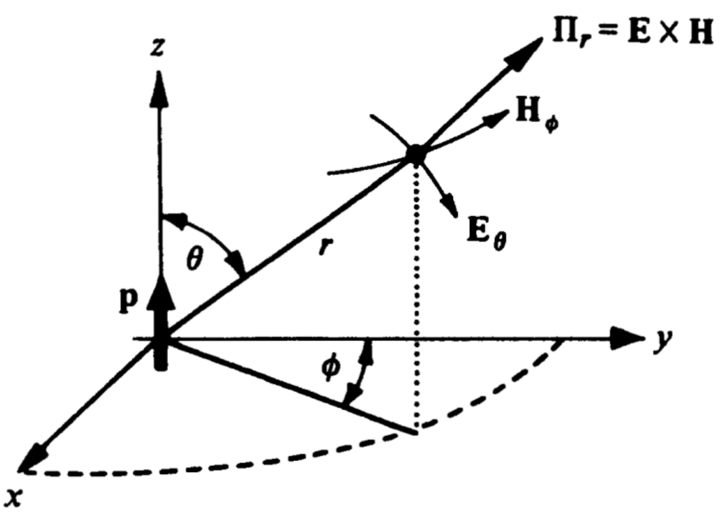


Fig. 4.18. Radiation from an oscillating electric dipole.

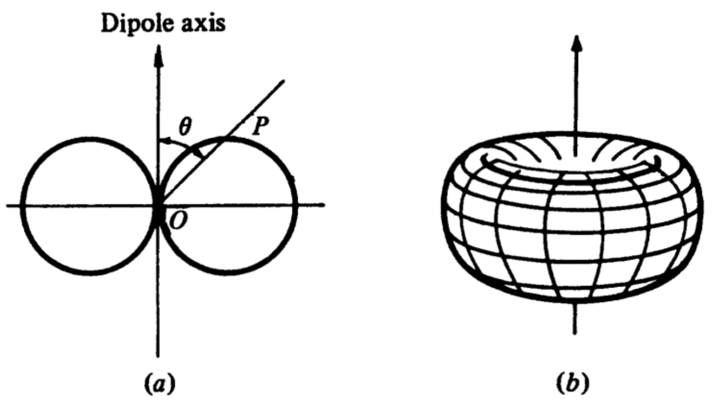


Fig. 4.19. Radiation polar-diagram for a dipole oscillator: (*a*) two-dimensional section; (*b*) three-dimensional sketch.

Notes

(i) Wave is of form of spherical wave, modulated by $\sin\theta$ (amplitude ~$1/r$, power ~$1/r^2$)

(ii) Power proportional to square of dipole moment

(iii)Power proportional to fourth power of frequency ω

The total power radiated by the dipole averaged over an optical cycle is

$$\langle P_0 \rangle = \frac{\omega^4 \rho_0^2}{12\pi\varepsilon_0 c^3}$$

Radiative decay

The most immediate consequence of the above formula is the phenomenon of radiative decay. Suppose an atomic oscillator is at form $t = t_0$ set oscillating with an amplitude P_0, but no further power is used to drive the oscillator. The dipole will radiate power, and consequently, by conservation of energy its amplitude p must decay in time.

If we write $p = ez$, then the energy in the oscillating dipole is

$$U(t) = \frac{1}{2}m\omega_0^2 z(t)^2 + \frac{1}{2}m\dot{z}(t)^2$$

The energy can be shown (exercise) to decay in time as

$$U(t) = U(t_0)e^{-(t-t_0)/\tau}$$

Where τ is the "lifetime" of the oscillator

The radiative decay rate within this purely classical model can be shown to be

$$\boxed{\gamma_{rad} = \frac{1}{\tau} = \frac{e^2\omega_0^2}{6\pi\varepsilon_0 mc^3}}$$

Plugging in the electron mass and visible frequencies, you find $\gamma_{rad} \simeq 10^8 s^{-1}$, or a lifetime of ~10 ns

Armed with these results, we can now consider an electromagnetic wave incident on a single atom, which we assume has a resonant frequency ω_0. The field will induce a polarization in the atom, and the atom will therefore radiate.

Def. mass +charge m_e, -e for electron ; m_n, +e for nucleus

$\vec{r}_{en} = \vec{r}_e - \vec{r}$ = relative position vector

$\vec{F}_{ne} = -\vec{F}_{e}$ $\vec{F}_{en}$ = force on e^- due to n (i.e. Harmonic restoring force)

The e.m. field exerts a force on each charge

$$\vec{F} = q(\vec{E} + \vec{V} \times \vec{B}) \simeq q\vec{E}$$

Since the $\vec{V} \times \vec{B}$ term is negligible at nonrelativistic speeds (is smaller by V/c => negligible for optical fields in linear response regime. It is important in experiments with terawatt lasers, but in that regime the Lorentz model is nonsensical

Newton :

$$m_e \frac{d^2\vec{r}_e}{dt^2} = -e\vec{E}(\vec{r}_e, t) + \vec{F}_{en}(\vec{r}_{en})$$

$$m_n \frac{d^2\vec{r}_n}{dt^2} = e\vec{E}(\vec{r}_n, t) + \vec{F}_{ne}(\vec{r}_{en})$$

Define center-of-mass coordinates:

$$\vec{R} = \frac{m_e\vec{r}_e + m_n\vec{r}_n}{m_e + m_n}; \vec{x} = \vec{r}_{en}$$

$$\Rightarrow \quad \vec{r}_e = \vec{R} + \frac{m_n}{M}\vec{x} \qquad M = m_e + m_n$$

$$\vec{r}_n = \vec{R} - \frac{m_e}{M}\vec{x}$$

Substitute these into Newton's equation:

$$m_e \frac{d^2\vec{R}}{dt^2} + m\frac{d^2\vec{x}}{dt^2} = -e\vec{E}(\vec{R} + \frac{m_n}{M}\vec{x}, t) + \vec{F}_{en}(\vec{x})$$

Where $m = \frac{m_e m_n}{m_e + m_n}$ = reduced mass

$$m_n \frac{d^2\vec{R}}{dt^2} - m\frac{d^2\vec{x}}{dt^2} = e\vec{E}(\vec{R} - \frac{m_e}{M}\vec{x}, t) + \vec{F}_{ne}(\vec{x})$$

$$m_n \frac{d^2\vec{R}}{dt^2} - m\frac{d^2\vec{x}}{dt^2} = e\vec{E}(\vec{R} - \frac{m_e}{M}\vec{x}, t) - \vec{F}_{en}(\vec{x})$$

We can add and subtract these two equations to separate $\vec{R}$ and $\vec{x}$:

Add =>

$$M\frac{d^2\vec{R}}{dt^2} = e[\vec{E}(\vec{R} - \frac{m_e}{M}\vec{x}, t) - \vec{E}(\vec{R} + \frac{m_n}{M}\vec{x}, t)]$$

Subtract => $$(m_e - m_n)\frac{d^2\vec{R}}{dt^2} + 2m\frac{d^2\vec{x}}{dt^2} = -e[\vec{E}(\vec{R} - \frac{m_e}{M}\vec{x}, t) + \vec{E}(\vec{R} - \frac{m_e}{M}\vec{x}, t)] + 2\vec{F}_{en}(\vec{x})$$

Consider the J^{th} component of the field, and Taylor expand to first orders:

$$E_i(x + \Delta x) = E_i(x) + \frac{dE_i(x)}{dx}\Delta x$$

Now in 3-D, the slope along x is $\hat{x} \cdot \nabla\vec{E}$, so in general we have

$$E_i(\vec{R}+\Delta\vec{r}) \simeq E_i(\vec{R}) + (\sum_i \Delta r_i \cdot \hat{x}_i) \cdot \nabla E_i$$

This can be written

$$\vec{E}(\vec{R}+\Delta\vec{r}) \simeq E(\vec{R}) + \Delta\vec{r} \cdot \nabla\vec{E}$$

Thus
$$\vec{E}(\vec{R} \pm \frac{m_i}{M}\vec{x}, t) \simeq \vec{E}(\vec{R}, t) \pm \frac{m_l}{M}\vec{x} \cdot \nabla\vec{E}(\vec{R}, t)$$

The justification for keeping only the first term is that for optical fields $\lambda \simeq$ 5000 Å but $x \simeq$ 1Å, so the approximation is equivalent to the <u>dipole approximation</u>.

The center-of-mass eqn. now becomes

$$M\frac{d^2\vec{R}}{dt^2} = \mathrm{e}[-\frac{m_e}{M}\vec{x} \cdot \nabla\vec{E}(\vec{R}, t) - \frac{m_n}{M}\vec{x} \cdot \nabla\vec{E}(\vec{R}, t)]$$

Or
$$\boxed{M\frac{d^2\vec{R}}{dt^2} = -\mathrm{e}\vec{x}\nabla\vec{E}(\vec{R}, t)}$$

The center of mass of the atom is thus accelerated by the <u>gradient</u> of the field

This force exerted by spatially varying fields is actually quite useful for trapping and cooling atoms.

In fact, we shall see shortly that the dipole moment of a Lorentz atom is proportional to the applied field, so $\vec{P} = -e\vec{x} = \alpha\vec{E}$

Where the proportionality constant α is called the <u>polarizability</u>.

Then
$$M\frac{d^2\vec{R}}{dt^2} = \vec{P} \cdot \nabla\vec{E} = \alpha\vec{E} \cdot \nabla\vec{E} = \frac{\alpha}{2}\nabla(\vec{E} \cdot \vec{E})$$

But
$$\nabla(\vec{E} \cdot \vec{E}) = \vec{E} \cdot \nabla\vec{E} + \left(\nabla\vec{E}\right) \cdot \vec{E} = 2\vec{E} \cdot \nabla\vec{E}$$

Or
$$\boxed{\vec{F} = \frac{\alpha}{2}\nabla|E|^2}$$

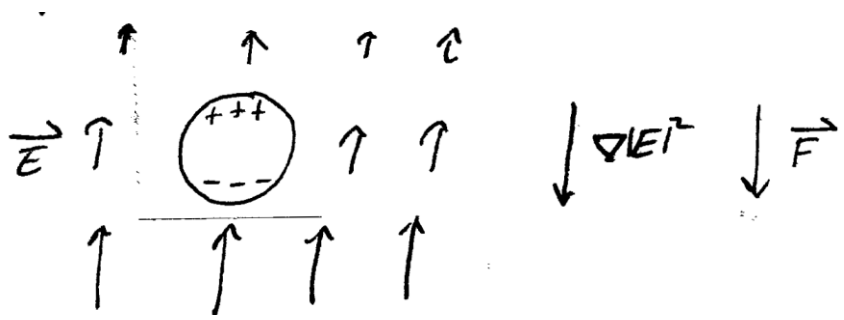
Which is the force on the center of mass of the atom .It just follows the gradient of the intensity!

Now $\vec{P} = \alpha\vec{E}$ for any kind of small (sub-λ) dielectric particle in a weak electric field

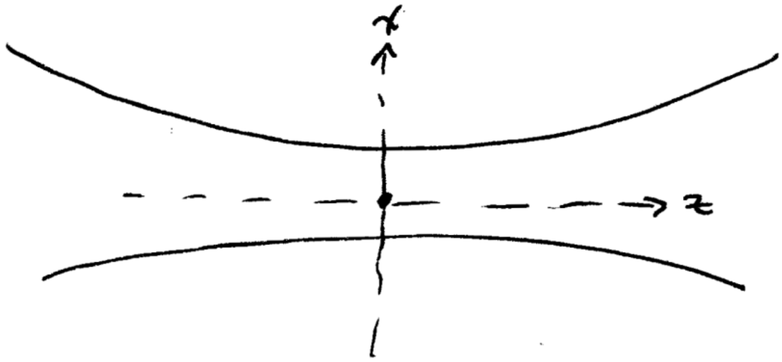
E,g. For a dielectric sphere with index n in a medium with index n_m, def. $n_r = n / n_m$

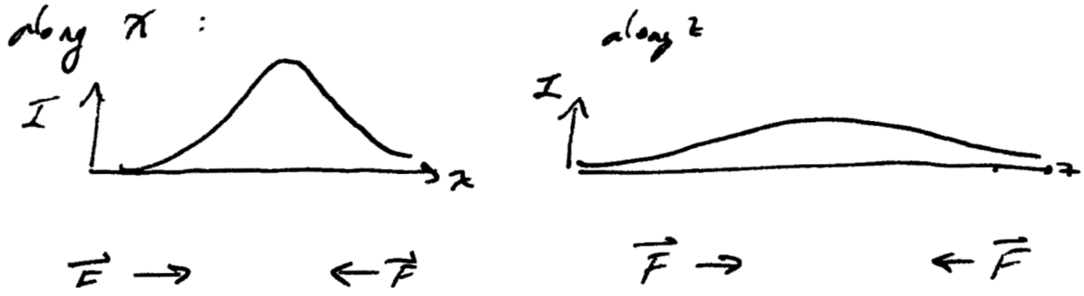
$$\alpha = n_m{}^2 a^3 (\frac{n_r{}^2 - 1}{n_r{}^2 + 2})$$

So if $n_r > 1, \alpha > 0$, and this particle can be trapped



E.g. Gaussian beam at focus





“<u>Optical tweezers</u>”