Lecture 17

As a result of expressions like this, it is clear that it can be very useful, then, to consider the possibility of the angular frequency being considered to be a complex variable, rather than the real variable we have always considered it to be.

In other words, we can write a field

$$E(t) = E_0 e^{i\omega t}$$

Where now, not only are E and E_0 complex (which we do in order to account for phase), but ω is complex too. This extension of the definition of E is often called the "complex analytic signal" associated with the physical electric field.

Note how damping naturally comes in this formulation. Decompose ω into its real and imaginary parts.

$$\omega = \omega_r + i\Gamma$$

=> $$E(t) = E_0 e^{i(\omega_r + i\Gamma)t} = E_0 e^{-\Gamma t} e^{i\omega_r t}$$

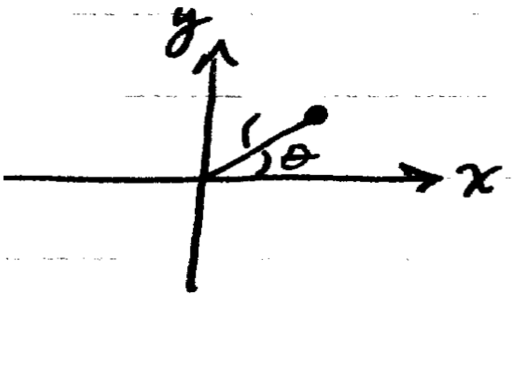
As usual, the physically significant (i.e. measurable) part of E is the real part

$$\mathrm{Re}(E) = (\mathrm{Re}\, E_0) e^{-\Gamma t} \cos(\omega_r t + \phi)$$

Clearly the real part of ω corresponds to the frequency of the harmonic oscillation, and the imaginary part corresponds to the temperal damping of the field. (or polarization)

Now that we are free to consider ω as a complex variable, we can go back to our integral (*) p.114. It will be instructive to use complex-variable integration methods to evaluate the integral.

Digression: A review of some important results in complex-function theory.



Complex variable $z = x + iy = re^{i\theta}$

Complex function f(z)

Derivative: as usual

$$\frac{df}{dz} = \lim_{\Delta z \to 0} \frac{f(z + \Delta z) - f(z)}{\Delta z} = f'(z)$$

If the derivative exists in a region R of the complex plane, f is said to be analytic in R.

$f(z_0)$ is analytic at z_0 if there is neighborhood $|z - z_0| < \delta$ at all points of which $f'(z)$ exists.

Quite often, functions we are interested in fail to be analytic in a very specific way.

A point at which f fails to be analytic is called a singularity.

z_0 is called an isolated singularity of f if we can find $\delta > 0$ such that the circle $|z - z_0| = \delta$ encloses no singular point other than z_0

A specific functional form containing an isolated singularity is $f(z) = \dfrac{g(z)}{z - z_0}$ Where $g(z)$ is analytic in a region containing z_0

A function of this form is said to have a simple pole of $z = z_0$

Def.: if $\lim\limits_{z \to \Delta z} (z - z_0)^n f(z) = A \neq 0$,

then $z = z_0$ is called a pole of order n.

ex. $f(z) = \dfrac{1}{(z - z_0)\left(z - z_0'\right)^3}$

has a simple pole at z_0 and a third order pole at $z = z_0'$

It should be clear why functions like this are interesting in linear response theory. Look at the integrand in (*) p.114.

$$\frac{e^{i\omega t}}{(\omega - \omega_1)(\omega - \omega_2)}$$ has simple poles at ω_1 and ω_2.

Physical significance of the poles:

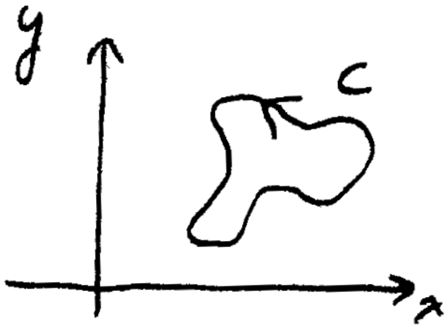
Note that ω_1 and ω_2 are the (positive and negative) natural oscillation! (Go back and look at our free-field solution on P.81)

This is a general result: the poles of the response function gives the resonant frequencies of the system.

Back to mathematics:

If $f(z)$ is analytic, the limit $f'(z) = \lim\limits_{\Delta z \to 0} \dfrac{f(z + \Delta z) - f(z)}{\Delta z}$ exist .This is a much more restrictive condition than it is for real variables. The reason is that the limit must exist (and be the same) no matter which direction you approach the point z in the complex plane.

We will not prove the following result, but the main consequence of the analyticity condition turns out to be Cauchy’s theorem:



Suppose g(z) is analytic inside and on a simple closed curve C. Then

$$\oint_c g(z)dz = 0$$

Now let’s consider integrals of functions with a simple pole.

Cauchy’s integral formula

If f(z) is analytic inside and on a simple closed curve C and z_0 is any point inside C, then

$$f(z_0) = \frac{1}{2\pi i}\oint_c \frac{f(z)}{z - z_0}dz$$

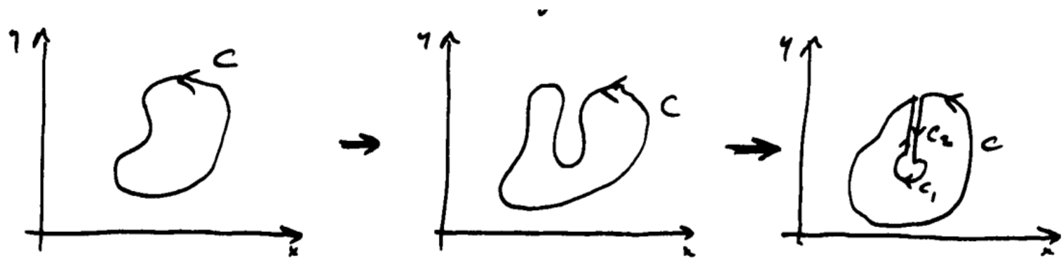
Where C is traversed in the positive (counter clockwise) sense

Note that the function $f(z)/(z - z_0)$ has a simple pole of z_0 .

Exercise for the reader: you should be able to show that Cauchy’s theorem is a simple special case of this.
Heuristic “derivation” of the integral formula:

First, let’s consider a consequence of Cauchy’s Theorem.
If the integrand is analytic, then the integration curve C can be deformed (within the region of analyticity) and one still has $\oint_c g(z)dz = 0$.

This may be seen by deforming the curve as follows



Clearly the integral along C_2 "down" cancels the integral along C_2 "up", so

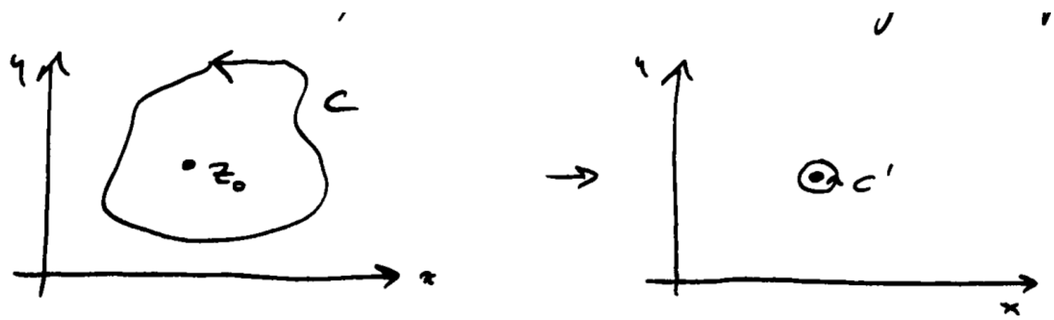
$$\oint_c g(z)dz = -\oint_{c_1} g(z)dz$$

In fact, one can smoothly deform C without changing the value of the integral as long as the deformation does not cross any singularities of g(z).
Now consider

$$g(z) = f(z)/(z - z_0)$$

Where f(z) is analytic everywhere (so g(z) has one simple pole).

Thus we can transform the integral $\oint_c g(z)dz = 0$ around an arbitrary curve C which endorses the pole to an integral around an infinitesimally small circle surrounding the pole :



$$\oint_c g(z)dz = \oint_{c'} g(z)dz$$

Now, if f(z) is a sufficiently "smooth" function, then in the infinitesimally small region around z_0 defined by C' ,its value will be essentially constant, so

$$\oint_{c'} \frac{f(z)}{(z - z_0)} dz \simeq f(z_0) \oint_{c'} \frac{dz}{(z - z_0)}$$

By a change of variables $z' = z - z_0$, $dz = dz'$, we have

$$f(z) \oint_{c'} \frac{dz}{(z - z_0)} = f(z_0) \oint_{c'} \frac{dz'}{z'}$$

Write $z' = \varepsilon e^{i\theta}$ where ε = radius of curve C'

$$dz' = i\varepsilon e^{i\theta} d\theta$$

$$f(z_0)\oint_{c'} \frac{dz'}{z'} = f(z_0)\int_0^{2\pi} \frac{i\varepsilon e^{i\theta} d\theta}{\varepsilon e^{i\theta}} = if(z_0)\int_0^{2\pi} d\theta = 2\pi i f(z_0)$$

Thus we find the Cauchy integral formula

$$f(z) = \frac{1}{2\pi i}\oint_c \frac{f(z)}{z - z_0} dz$$

Again, this is a heuristic derivation, not a proof, but the result is straightforwardly proved in general.

Definition: If $g(z) = \dfrac{f(z)}{z - z_0}$ where f(z) is analytic, then $f(z_0) = \left(z - z_0\right) g(z_0)$ is called the

<u>residue</u> of g at z_0 .

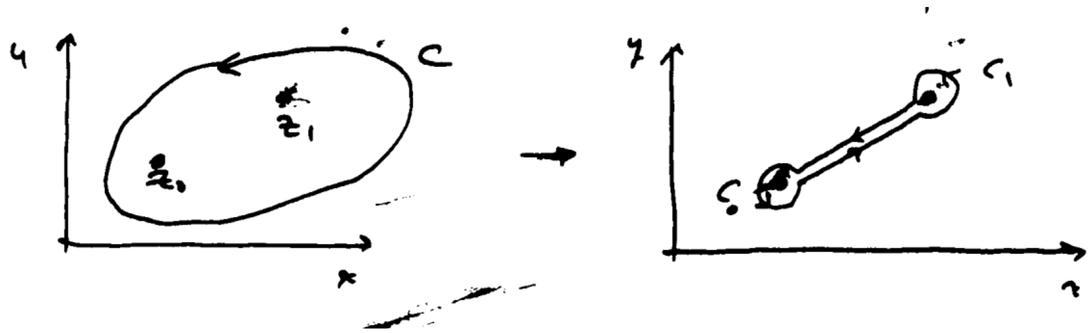
<u>Cauchy's integral formula</u> is thus often recast as

$$\boxed{\oint_c g(z)dz = 2\pi i \,\mathrm{Res}\, g(z_0)}$$

If the integrand has multiple poles inside C, e.g.

$$g(z) = \frac{f(z)}{(z - z_0)(z - z_1)}$$

Then since g is analytic everywhere except at z_0 , z_1 ,C can be deformed as we have seen previous .



The integrations along the straight lines between the poles cancel (since the directions are opposite), so

$$\oint_c = \oint_{c_1} + \oint_{c_2}$$

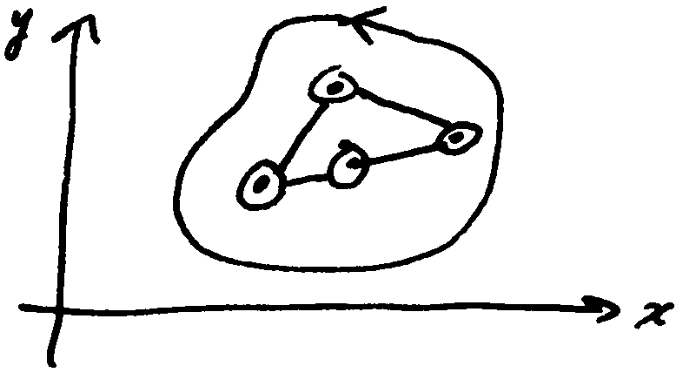
Thus

$$\oint_c g(z)dz = 2\pi i \mathrm{Res}\, g(z_0) + 2\pi i \mathrm{Res}\, g(z_1) = \frac{2\pi i\, f(z_0)}{z_0 - z_1} + \frac{2\pi i\, f(z_1)}{z_1 - z_0}$$

In general, if g has multiple poles within C

$$\oint_c g(z)dz = 2\pi i \sum \mathrm{Residues}$$

The proof is an extension of the argument above

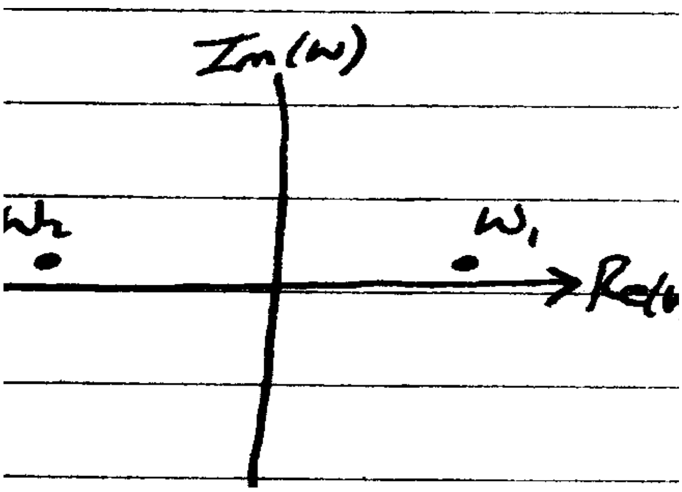


This is actually all the complex-variable calculus we will need. We are trying to evaluate

$$\chi(t) = -\frac{\omega_p^2}{2\pi}\int_{-\infty}^{\infty}\frac{e^{i\omega t}}{(\omega-\omega_1)(\omega-\omega_2)}d\omega$$

Where $\quad \omega_{1,2} = i\frac{\gamma}{2} \pm \sqrt{\omega_0^2 - \gamma^2/4}$

The integral has two simple poles in the <u>upper half plane</u>



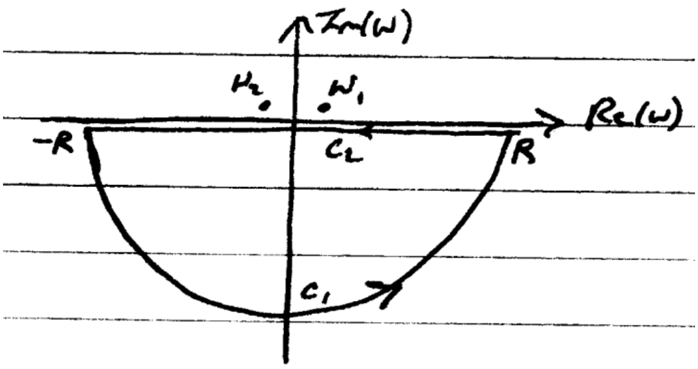
Note that the integral we want is <u>along the real</u> axis from $\pm\infty$, i.e.

$$\lim_{R\to\infty}\int_{-R}^{R}\frac{e^{i\omega t}}{(\omega-\omega_1)(\omega-\omega_2)}d\omega$$

We can evaluate the integral using Cauchy's integral formula if we pick the curve C wisely.

(1) Consider times <u>t<0</u>

Choose C to be a semicircle in the lower half plane: $C = C_1 + C_2$



Why choose this curve ? $\quad \oint \;= \oint_{C_1} + \oint_{C_2}$

a) the integral we want is from –R to R along the real axis (with a minus sign from the direction)

b) along the semicircle, $\omega = Re^{i\theta}$, where θ runs from π to 2 π

R is large ($\to\infty$) so $|R| \gg |\omega_1|, |\omega_2|$

$$\Rightarrow \int_{C_1} = \lim_{R\to\infty} \int_{\pi}^{2\pi} \frac{e^{i(Re^{i\theta})t}}{R^2}\left(d\theta iRe^{i\theta}\right)$$

$$|\int_{C_1}| < \lim_{R\to\infty} \int_{\pi}^{2\pi} \frac{|e^{i(Re^{i\theta})t}|}{R} d\theta$$

$$|e^{iRe^{i\theta}t}| = |e^{iR(\cos\theta + i\sin\theta)t}| = e^{-R\sin\theta\, t}$$

Now, when $\pi < \theta < 2\pi, -1 < \sin\theta < 0$

=>when t<0, $e^{-R\sin\theta\, t} \to 0$ as $R\to\infty$

Thus $\oint_{c_1} = 0$

By Cauchy's Theorem $\oint_c = 0$ also, since there are no poles inside C

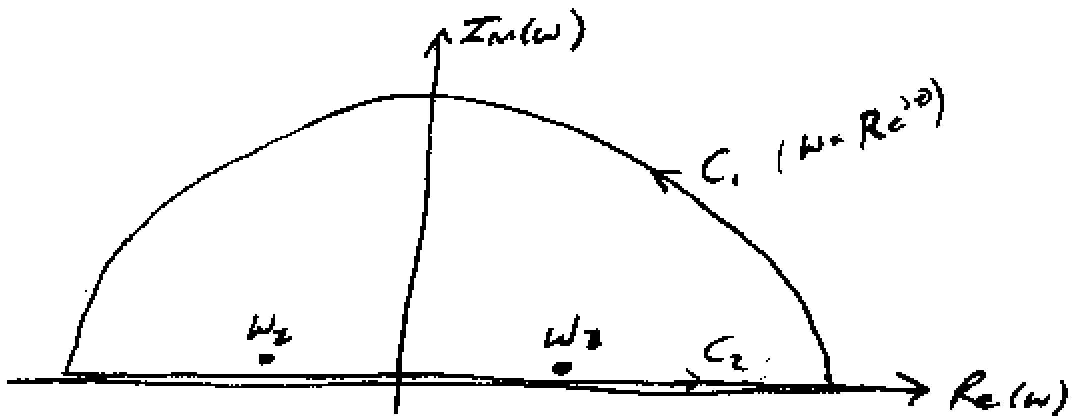
Therefore $\oint_{c_1} = 0$ also

Therefore $\boxed{\chi(t) = 0 \text{ for } t < 0}$

This is exactly what causality requires!
The response function must be <u>zero</u> for negative times, and our harmonic oscillator solution has now been explicitly shown to satisfy this condition.

(2) consider times t> o

Clearly, we expect a nonzero result here. Now, we choose our curve in the <u>upper</u> half phase:



Now

$$\oint_c = 2\pi i \sum \operatorname{Re} sidues$$

$$\int_{-\infty}^{\infty} \frac{e^{i\omega t}}{(\omega-\omega_1)(\omega-\omega_2)} d\omega + \int_{C1} = \left[\frac{e^{i\omega_1 t}}{\omega_1-\omega_2} + \frac{e^{i\omega_2 t}}{\omega_2-\omega_1} \right]$$

Using the same argument as before, (proof left for reader)

$$\int_{C1} \to 0 \quad \text{as} \quad R \to \infty$$

=>

$$\int_{-\infty}^{\infty} \frac{e^{i\omega t}}{(\omega-\omega_1)(\omega-\omega_2)} d\omega = 2\pi i \frac{e^{i(\omega_1+\omega_2)t/2}}{\omega_1-\omega_2} \left[e^{i(\omega_1-\omega_2)t/2} - e^{-i(\omega_1-\omega_2)t/2} \right] = -4\pi \frac{e^{i(\omega_1+\omega_2)t/2}}{\omega_1-\omega_2} \left[\frac{e^{i(\omega_1-\omega_2)t/2} - e^{-i(\omega_1-\omega_2)t/2}}{2i} \right]$$

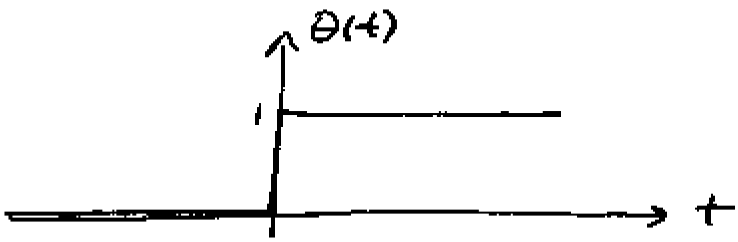
$$\omega_1 - \omega_2 = \left(i\frac{\gamma}{2} + \sqrt{\omega_0^2 - \gamma^2/4} \right) - \left(i\frac{\gamma}{2} - \sqrt{\omega_0^2 - \gamma^2/4} \right) = 2\sqrt{\omega_0^2 - \gamma^2/4} = 2\upsilon_0$$

And $\quad \dfrac{\omega_1+\omega_2}{2} = i\dfrac{\gamma}{2}$

-> $\quad \chi(t) = \dfrac{\omega_p^2}{2\pi} \cdot 4\pi \cdot \dfrac{e^{-\gamma t/2}}{2\upsilon_0} \sin \upsilon_0 t, t > 0$

$$\boxed{\chi(t) = \frac{\omega_p^2 \sin \upsilon_0 t}{\upsilon_0} e^{-\gamma t/2} \theta(t)}$$

Where $\theta(t)$ =unit step function



Note that:

(1) $\theta(t)$ guarantees causality

(2) Damped oscillation (includes renormalized frequency)

(3) amplitude proportional to $Ne^2/\varepsilon_0 m$ as in frequency domain

(4) it's a sine oscillation , not a cosine !

Of course, now that we have $\chi(t)$, the time-dependent polarization can be found for an arbitrary driving field E(t) via the convolution integral .