# Lecture 6 Propagation in a conducting medium

So far we have considered propagation only in a uniform “lossless dielectric”, where we have $\vec{D} = \varepsilon \vec{E}$ and $\vec{B} = \mu \vec{H}$ with $\varepsilon$ and $\mu$ being <u>real constants.</u>

In a conducting medium we must also take into account the current that can be induced in the medium by the electric field of an e.m. wave .We assume the simplest case, which is $\underline{\vec{J} = \sigma \vec{E}}$ ($\sigma$ =<u>conductivity</u> of the medium).

Now Maxwell’s equations are

$$\nabla \cdot \vec{E} = 0 \qquad \nabla \cdot \vec{B} = 0$$

$$\nabla \times \vec{E} = -\frac{\partial \vec{B}}{\partial t} \qquad \nabla \times \vec{H} = \sigma \vec{E} + \varepsilon \frac{\partial \vec{E}}{\partial t}$$

The derivation of the wave eqn. goes exactly as before (taking $\nabla \times \nabla \times \vec{E}$ ), but we get an extra term: $\nabla^2 \vec{E} = \mu\varepsilon \frac{\partial^2 \vec{E}}{\partial t^2} + \mu\delta \frac{\partial \vec{E}}{\partial t}$

Such an equation, with both first and second time derivations, is known as a “telegraph equation” We shall see below that the first-time-derivative results in <u>damping</u> of the wave.

**<u>Harmonic waves:</u>**

Consider solutions of the form

$$\vec{E} = \vec{E}(\vec{r})e^{i\omega t}$$

The wave equation then becomes

$$\nabla^2 \vec{E}(\vec{r}) = -\omega^2 \mu\varepsilon \vec{E}(\vec{r}) + i\omega\mu\sigma \vec{E}(\vec{r})$$

Or

$$\nabla^2 \vec{E}(\vec{r}) + \omega^2 \mu(\varepsilon - \frac{i\sigma}{\omega})E(\vec{r}) = 0$$

This looks just like the wave equation we had before, but now it behaves as if the dielectric constant is complex.

Def. $\tilde{\varepsilon} = \varepsilon - \dfrac{i\sigma}{\omega}$ or $\tilde{\varepsilon}_r = \varepsilon_r - \dfrac{i\sigma}{\omega\varepsilon_o}$ (dimensionless form) complex dielectric constant

=> $\nabla^2 \vec{E} + \omega^2 \mu\tilde{\varepsilon}\vec{E} = 0$ Complex Helmholtz eqn.

We can still write this in the familiar form of the Helmholtz equation, but now the wave vector must be complex:

$$\nabla^2 \vec{E}(\vec{r}) + \tilde{k}^2 \vec{E}(\vec{r}) = 0$$

Where $\tilde{k} = \omega\sqrt{\mu(\varepsilon - \frac{i\sigma}{\omega})}$

If we want to write $\tilde{k}$ in terms of an index of refraction, then it must also be complex:

$$\tilde{k} = \frac{\tilde{n}\omega}{c}$$

Where $\tilde{n}$ can be written in terms of its real and imaginary parts as

$$\boxed{\tilde{n} = n(1 - i\kappa)}$$

Where $\kappa$ is called the "extinction coefficient"

To see the consequences of a nonzero conductivity or imaginary part of the dielectric constant, consider a plane wave propagating in the z-direction.

$$\vec{E}(\vec{r}) = \vec{E_o} e^{-i\tilde{k}z}$$

$$= \vec{E_o} e^{-i\frac{n\omega}{c}(1-i\kappa)z} = \vec{E_o} e^{-i\frac{n\omega}{c}z} e^{-\frac{n\omega}{c}\kappa z} = \vec{E_o} e^{-i\frac{n\omega}{c}z} e^{-z/d}$$

$\vec{E_o} e^{-i\frac{n\omega}{c}z}$ : the usual harmonic wave

$e^{-\frac{z}{d}}$ : Exponentially damped amplitude

$d = \frac{c}{n\omega\kappa}$ ="skin depth" of the metal

Some straightforward algebra (e.g, Guenther, P.52) will allow n and $\kappa$ to be expressed in terms of $\varepsilon$, $\mu$, and $\sigma$ :

$$n^2 = \frac{c^2}{2}\left[\sqrt{\mu^2\varepsilon^2 + \left(\frac{\mu\sigma}{\omega}\right)^2} + \mu\varepsilon\right]$$

$$n^2\kappa^2 = \frac{c^2}{2}\left[\sqrt{\mu^2\varepsilon^2 + \left(\frac{\mu\sigma}{\omega}\right)^2} - \mu\varepsilon\right]$$

For typical metals, $\frac{\sigma}{\omega} \gg \varepsilon$ ,so $n^2\kappa^2 \simeq \frac{c^2\mu\sigma}{2\omega}$

And thus

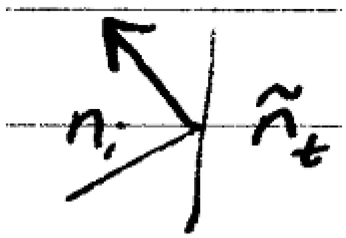
$$d \simeq \sqrt{\frac{2}{\mu\sigma\omega}}$$ (Equivalent to lipson's $d = l = \sqrt{\frac{2\varepsilon_o c^2}{\sigma\omega}}$ using $c^2 = \frac{1}{\varepsilon_o\mu_o}$)

i.e. the skin depth goes as $\frac{1}{\sqrt{\sigma}}$ and $\frac{1}{\sqrt{\omega}}$

In the visible (optical) region of the spectrum, the skin depth of typical metals is on the order of 10 Å

It should be noted that this is the simplest possible model for propagation in a conducting medium. A complete model of the frequency dependence of the propagation requires a more sophisticated model of the response of the metal to the applied field to obtain $\sigma(\omega)$.The color of the metal such as gold and copper can only be accounted for by a quantum-mechanical calculation of the energy band structure of the metal. Further treatments may be found in Born+ Wolf chap14, or any text on solid state physics.

**<u>Reflection from a conductor</u>**




You will notice that the main assumptions we made in deriving the Fresnel equations for reflection from a dielectric wave that (i)the material response is linear, and (ii) $\varepsilon$ and $\mu$ are constant on each side of the interface. The <u>boundary conditions</u> and thus the <u>reflectivity formulas</u> therefore hold just as well for a complex dielectric constant as for a real one.

Thus the Fresnel field reflectivity's given above (P, 30, 32) apply, but $\theta_t$ is complex

$$\sin\tilde{\theta}_t = \frac{n_i}{\tilde{n}_t}\sin\theta_i$$

The fact that the reflection coefficient becomes complex means there is a phase shift between incident and reflected waves. Thus linearly polarized light can become elliptically polarized on reflection under certain circumstances (see Born+Wolf for a discussion)
We will not concern ourselves with the details of the general case, but restrict our attention solely to power reflection at normal incidence.

$$R = |r|^2 = \frac{\tilde{n}-1}{\tilde{n}+1}\cdot\frac{\tilde{n}^*-1}{\tilde{n}^*+1} = \frac{(n-1)^2+(n\kappa)^2}{(n+1)^2+(n\kappa)^2} = 1-\frac{4n}{(n+1)^2+(n\kappa)^2}$$

Note that if the index were purely imaginary, i.e. $\tilde{n} = in\kappa$, then we would have

$$R = \frac{(in\kappa-1)(-in\kappa-1)}{(in\kappa+1)(-in\kappa+1)} = 1$$

=> Perfect reflector

Indeed, the imaginary part of the dielectric constant does dominate over the real part, and very high reflectivity (70~95%) are observed for most (good) metals.