Lecture 19

Pulse Propagation

We have seen above that, in general, wave groups or pulses propagate at the group velocity $d\omega/\mathrm{dk}$ (see Siegman lasers section 9.1 for an explicit treatment of Gaussian pulse propagation. On the other hand we have send nothing about the shape of the pulse and how it might change with propagation in the dispersive media.

The general approach is straight forwardly taken from linear response theory.

1. Given $\mathrm{E}(z_i,t)$ at the input to the linear system, decompose into its frequency components via Fourier transformation

$$\mathrm{E}(z_i,\omega)=\mathrm{F}[\mathrm{E}(z_i,t)]$$

2. Given the linear transfer function of the medium, propagate each frequency component through the medium:

$$\mathrm{E}(z,\omega)=\mathrm{R}(\omega)e^{-i\Psi(\omega)}\mathrm{E}(z_i,\omega)$$

Where $\mathrm{R}(\omega)$ = (real) amplitude response (describe linear loss)

$\Psi(\omega)$ = (real) phase response (describe dispersion)

3. Find the time-domain field at the output by inverse Fourier transformation.

$$\mathrm{E}(z,t)=\mathrm{F}^{-1}\left[\mathrm{E}(z,\omega)\right]$$

The main question that must therefore be answered for any optical system is: what is transfer function?

For a linear dielectric, we know that

$$\vec{P}(\omega)=\varepsilon_0\chi(\omega)\vec{\mathrm{E}}(\omega)$$

Lends to $\nabla^2\vec{\mathrm{E}}+\dfrac{\omega^2}{c^2}(1+\chi)\vec{\mathrm{E}}=0$

Or $\nabla^2\vec{\mathrm{E}}+\dfrac{\tilde{n}^2\omega^2}{c^2}\vec{\mathrm{E}}=0$

⇨ If we define as usual the complex propagation constant

$$\tilde{k}(\omega)=\frac{\tilde{n}(\omega)\omega}{c}$$

Then the transfer function is just $\boxed{e^{-i\tilde{k}(\omega)z}}$

Usually (as we have seen for example in the CEO model), $\tilde{k}(\omega)$ can be a very complicated function of frequency, making numerical solutions of the propagation problem necessary. Far from resonances, it is often possible to make a Taylor expression around the carrier frequency $\bar{\omega}$ of

the pulse:

$$\tilde{k}(\omega) \simeq k(\bar{\omega}) + k'(\bar{\omega})(\omega - \bar{\omega}) + \frac{1}{L}\frac{d^2k}{d\omega^2}\Big|_{\bar{\omega}}(\omega - \bar{\omega})^2$$

$k(\bar{\omega})$: overall phase shift

$$\frac{dk}{d\omega}\Big|_{\bar{\omega}}(\omega - \bar{\omega}) \quad : V_g^{-1}$$

$$\frac{1}{L}\frac{d^3k}{d\omega^2}\Big|_{\bar{\omega}}(\omega - \bar{\omega})^2 \quad : \text{“group –velocity dispersion ” (responsible for pulse spreading)}$$

As this subject form the core of 546, we will have to leave it here.

Light Scattering

So far we have considered propagation of light

1 . in vacuum

2. across a single abrupt discontinuity

3. in a uniform linear dielectric medium

In all cases, the direction of propagation $\vec{k}$ was always well defined, and once we knew an initial value of $\vec{k}$, we could calculate where the wave would go.

On the other hand, unless you are foolish enough to look directly into the sun or some other light source, most of the light you see is scattered light! We have two basic goals here:

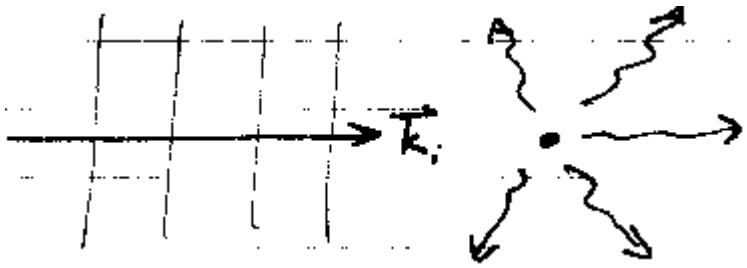
1, to understand the origin of light scattering

2, to understand its frequency dependence.

Before we start, we should note that in both cases 3 and 4 above, there was no light scattering. A plane wave propagating in a uniform medium remained a plane wave, and even in a nonuniform medium with slow variation of the index, there was no scattering– only a slow change in direction $\vec{k}$. We shall return to the significance of this point below!

1. Scattering by a single molecule

   This is the simplest possible case. We consider a plane wave of incident intensity I_{inc} or incident field strength E_{inc} impinging on a single atom or molecule, which as usual we consider to be a classical dipole oscillator (CEO model) is



We know from previous discussions that the field induces a dipole moment in the molecule

$$\vec{\mu} = \alpha(\omega)\vec{E}_{inc}\cos\omega t$$

Where $\alpha(\omega)$ is the polarizability of the molecule.

Remember that the polarizability in the CEO model is

$$\alpha(\omega) = \frac{e^2/m}{\omega_0^2 - \omega^2 + i\omega\gamma}$$

Showing that the magnitude of the induced dipole is large near resonance, but small far from resonance.

We also know (see P.75) that an oscillating dipole radiates power at a rate

$$W = \frac{\omega^4\mu^2}{6\pi\varepsilon_0 c^3}\cos^2\omega t = \frac{\omega^4\alpha^2(\omega)E^2{}_{mc}}{6\pi\varepsilon_0 c^3}\cos^2\omega t$$

It is useful to cast this in terms of the incident intensity

Poynting vector $\vec{S} = \varepsilon_0 c\hat{k}E^2{}_{mc}\cos^2\omega t \quad (\hat{k} = \frac{\vec{k}}{k})$

⇨ $$E^2{}_{mc}\cos^2\omega t = \frac{1}{\varepsilon_0 c}\left|\vec{S}\right|$$

Time –averaging as usual over one cycle gives

$$\frac{E^2{}_{inc}}{2} = \frac{1}{\varepsilon_0 c}I_{inc} \qquad I = \left\langle\left|\vec{S}\right|\right\rangle$$

⇨ $$\left\langle|W|\right\rangle = \left[\frac{\omega^4\alpha^2(\omega)}{12\pi\varepsilon_0 c^4}\right]I_{mc}$$

Note the units: W= power, $I_{mc} = \frac{power}{area}$

Thus the thing in brackets [] must have dimensions of area.

Thus we call it the scattering cross section of the atom:

$$\sigma(\omega) = \frac{\omega^4\alpha^2(\omega)}{12\pi\varepsilon_0 c^4}$$

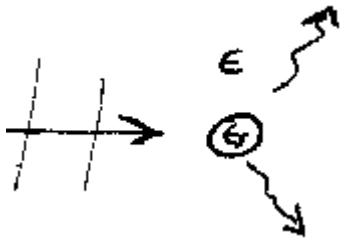
Note that the cross section is a property of a single atom. It is helpful to think of it as the "effective area" of an atom, such that any light incident within that area is scattered.

2. Scattering form a dielectric sphere

ε_s =dielectric constant of sphere

ε = dielectric constant of surrounding radium

a = radius of sphere



If $a \ll \lambda$, i.e. the sphere is smaller than a wavelength, then the polarizability of the sphere may be calculated using the laws of electrostatics. The result may be found in any E+M text (dipole approximation):

$$\alpha = \frac{\varepsilon_s - \varepsilon}{\varepsilon_s + 2\varepsilon} a^3$$

Note that any frequency dependence of α comes in through the frequency dependences of ε and ε_s.

This gives for the scattering cross section

$$\sigma = \frac{\omega^4 \alpha^6}{12\pi\varepsilon_0^2 c^4} \left(\frac{\varepsilon_s - \varepsilon}{\varepsilon_s + z\varepsilon} \right)^2$$

If $a \geq \lambda$, then the scattering can no longer be considered to be that of a point dipole. In this case, the shape of the particle becomes important for dielectric "blocks", and for sphere as well as arbitrary "blocks", all wavelengths $\lambda \leq a$ are scattered with roughly equal efficiency ($\sigma \sim$ constanrt).

3. Scattering from N molecules confined to a region $\ll \lambda$

If a collection of N sections are in a region $\ll \lambda$, then they will radiate in phase with each other, since the field that excites them $E = E_0 e^{-i\vec{k}\cdot\vec{r}}$ has the same phase for all the dipoles.

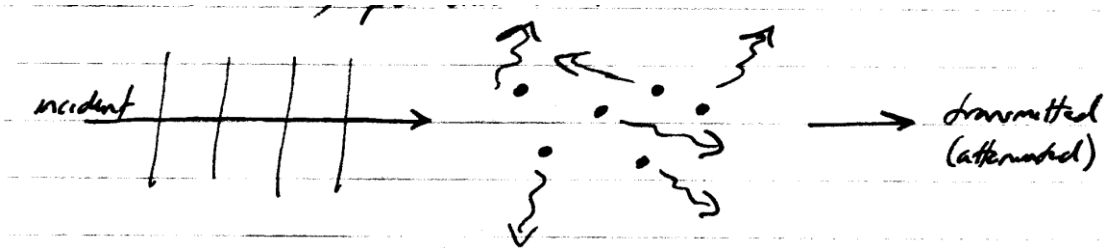
Therefore the total dipole moment of the "blob" is just $N\vec{\mu}$, where $\vec{\mu}$ is the moment of each dipole. Thus it is trivial to see that (*) p.142 becomes

$$\sigma(\omega) = \frac{\omega^4 N^2 \alpha^2(\omega)}{12\pi\varepsilon_0^2 c^4}$$

This is an extremely important result, known as "coherent scattering". Because the dipoles have the same phase, they are said to be coherent with respect to each other. Coherent radiation or coherent scattering always results in a scattering rate (or emission rate) which is N^2 times the rate for a single scatter of emitter. $\boxed{\sigma_{Ndipole} = N^2 \sigma_{1dipole}}$

Note that when the "blob" becomes $\geq \lambda$ in size, then the dipoles do not all have the same phase, and we have only $\boxed{\sigma_{Ndipole} = N * \sigma_{1dipole}}$ .we will return to this important point momentary.

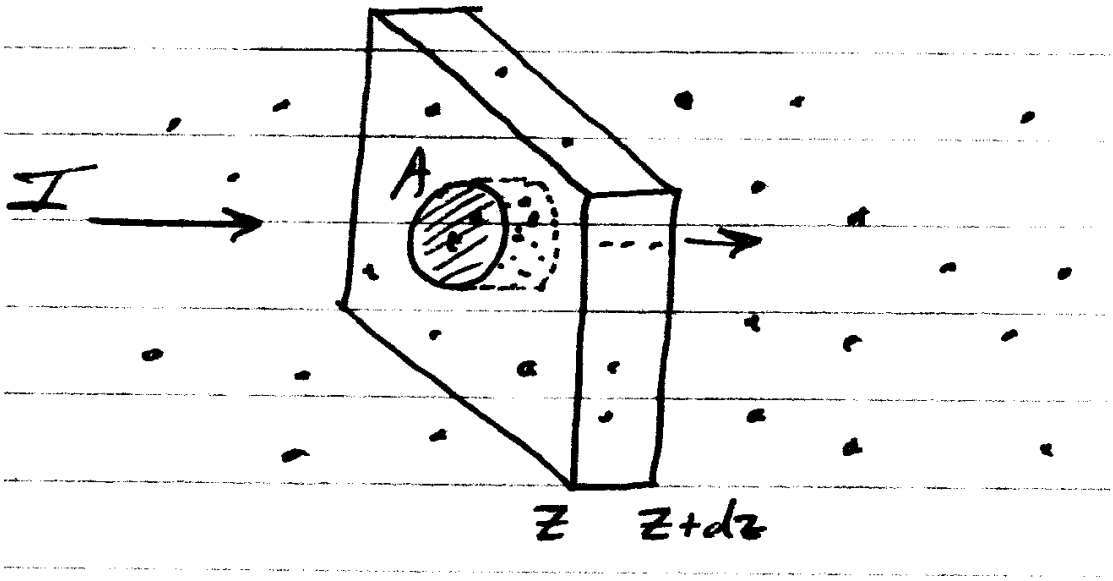
4. Scattering by a collection of independent scatters (size $\geq \lambda$)
   - Suppose there are N scatters (individual molecules or dielectric blobs) per unit volume.



Let us begin by taking the naïve point of view that each scattering center scatters power out of the incident wave independently of all the other scatters. Clearly the wave will be attenuated as it propagates, since it loses power to other propagation directions.

The attention of the wave may be calculated as follows.

Consider a beam of area A with an intensity $I(z)$ at position z



Power incident on plane at z is P(z)=AI(z). The power remaining in the beam of z+dz is P(z+dz).

The power scattered out of the beam is

#of atoms $\times$ scattered power per atom

$$P_{scatt} = (N \cdot A \cdot dz)\sigma I(z)$$

By conservation of energy we have

$$P(z) - P(z + dz) = \sigma N A$$

Or $\boxed{\frac{dI}{dz} = -\sigma N I}$ using $\frac{dI}{dz} = \lim_{dz \to 0} \frac{1}{A} \frac{P(z+dz) - P(z)}{dz}$

The solution to this equation is of course just

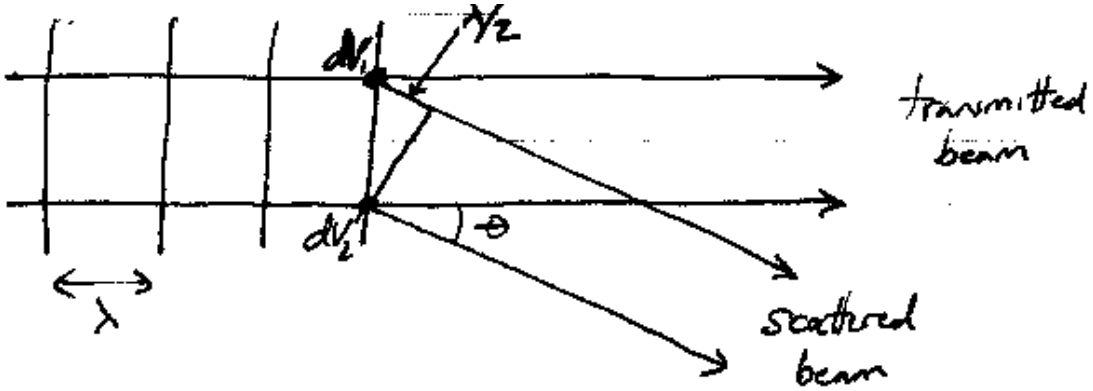
$$\boxed{I(z) = I_{mc} e^{-\sigma N z}}$$

Note that we get exponential attenuation, just as we did for absorption, although here no power is absorbed by the molecules – it is only scattered away.
The attenuation coefficient $\alpha = \sigma N$ is proportional to N (high density => rapid attenuation).

5. "Scattering " by a uniform distribution of scatters

It seems that the result we have just obtained, namely that atoms which don't absorb any light nevertheless cause attenuation of propagating waves by scattering, contradicts our previous results regarding wave propagation. Both the macroscopic electromagnet theory, which describes a dielectric constant to a dielectric medium and the CEO model, find that the only effect of the medium (which is made up, after all, if N atoms per unit volume) is to introduce the index of refraction. If the atoms don't absorb, the light is not attenuated, it merely propagates with a different velocity.

However, by considering in the previous section the radiation of light by individual dipoles, we found scattering and attenuation to occur. The question is: what is wrong?
Consider a plane wave incident on a perfectly uniform, homogeneous medium. (nonabsorbing):



We have labeled two scatters as volume elements dV_1 and dV_2^*.consider this scattering into an angle θ :

(i) For θ=0, clearly the phase difference between the two scattered waves is zero. Thus the scattering in the forward direction is coherent.

This is called, not surprisingly,"coherent forward scattering ".

(ii) Consider scattering from dV_1 at angle θ.

For any θ, another scattering element dV_2 can be found which radiates exactly out of phase in that direction (i.e. the path length difference is $\lambda/2$, as shown),if and only if the medium is perfectly homogeneous.

Conclusion: only coherent forward scattering occurs in a perfectly homogeneous medium! No light is scattered into directions $\theta \neq 0$.There is no attenuation!

Important note: the forward scattered wave may have some phase shift relation to the incident wave. Therefore the total wave in the forward direction, which is the sum of the incident (unscattered) wave plus the scattered wave, may lag or lead in phase of the incident wave. This phase difference is the physical origin of the index of refraction!

- Note: the size of dV_1 and dV_2 is $\ll \lambda$.

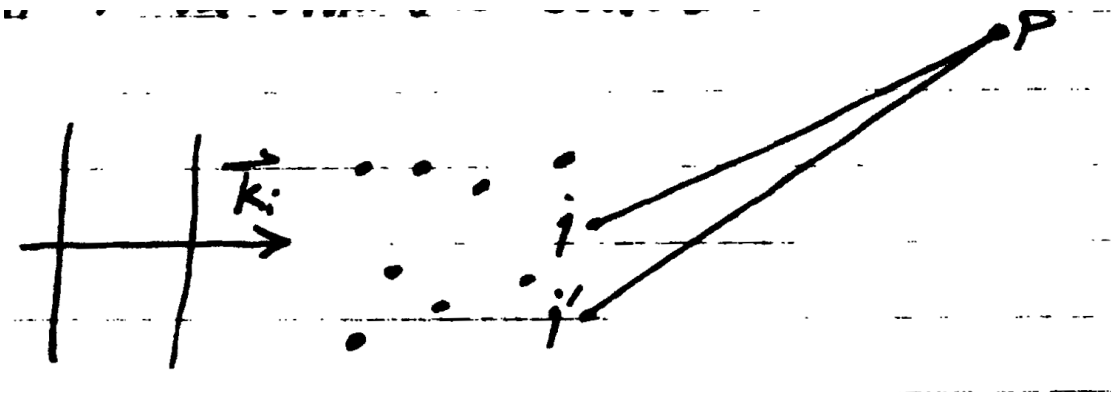
So, what was wrong with the argument given in 4 above?
Answer: we added intensities (or power), and not wave amplitude. This is wrong!
If you add wave amplitudes, then all the phase relationships between the dipoles are accounted for, and only forward scattering occurs.
This brings up the obvious question: when dose side scattering occur?

To answer this, we note that the scattering from dV_1 to dV_2 will not exactly equal to the number in dV_2, even though the optical path length difference is $\lambda/2$. The amplitudes of the two scattered waves are not equal, and therefore the cancellation is not complete.
Thus we see that scattering occurs only if the dielectric is not perfectly homogeneous. It is fluctuations in the optical properties of the medium that cause scattering.

Qualification: if the fluctuations occur on a length scale $\gg \lambda$, then side scattering again does not occur. This only results in a smooth variation in $\vec{k}$, as we saw in our treatment of the WKB problem.

6. Scattering by a random collection of N scatters
(with fluctuations on a length scale $\sim \lambda$)

Consider a plane wave incident on N scatters. We want to calculate the intensity at an observation point P far from the scatters.



The field at P due to the j^{th} dipole is

$$E_j = E_0 e^{-i\vec{k}\cdot\vec{r}_j + i\varphi_j}$$

Where E_0 is the field amplitude from one dipole.

The intensity at P is therefore proportional to

$$I_p = \left(\sum_{j=1}^{N} E_j\right)\left(\sum_{j'=1}^{N} E_{j'}^{*}\right) = \left|E_p\right|^2$$

$$I_p = \sum_{j,j'} \left|E_0\right|^2 e^{-i\vec{k}\cdot(\vec{r}_j - \vec{r}_{j'}) + i(\varphi_j - \varphi_{j'})}$$

The incident wave sets up the relative phases as

$$\varphi_j - \varphi_{j'} = \vec{k}_{i'}\cdot(\vec{r}_i - \vec{r}_{i'})$$