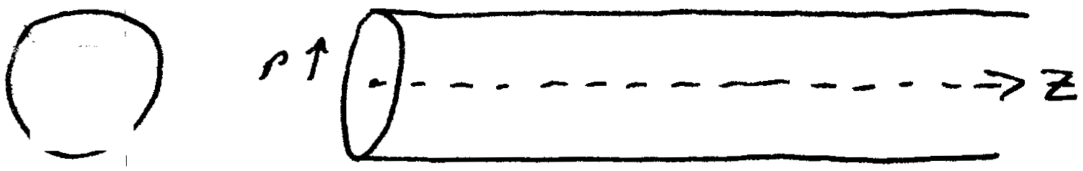
Lecture 23

Example: Graded-Index (GRIN) Optics

We will give a very brief treatment here of ray propagation in cylindrically symmetric GRIN media (e.g. GRIN fibers or GRIN lenses). Our goal is not to give comprehensive treatment GRIN optics, but to use it to illustrate how the general principles we have derived can be applied to practical problems. Much more detailed treatments can be found in Guenther chap.5 and Lizuka's Engineering Optics chap.5. We follow the treatment in Salch +Teich's Photonics $ 1.3

A graded-index medium has an index of refraction that varies in a controlled way such that it is a continuous function of position. The index can be controlled by adding impurity atoms to glass during fabrication. We consider the special case where the index is radially varying in a cylindrically symmetric medium.

$$n(\vec{R}) \to n(\rho)$$



We considered a similar problem in the context of WKB, but here we are not concerned with the form of the wave or with its phase variation, only with the ray trajectories.

In order to calculate ray trajectories, we need to solve the ray equation

$$\frac{d}{ds}(n\hat{s}) = \nabla n$$

We will specifically solve this equation for rays which propagate essentially along the z-axis (i.e. for rays at small angles with respect to the z-axis).

Then $\quad \hat{s} = \frac{d\vec{r}}{ds} = \frac{dx}{ds}\hat{x} + \frac{dy}{ds}\hat{y} + \frac{dz}{ds}\hat{z} \simeq \frac{dx}{dz}\hat{x} + \frac{dy}{dz}\hat{y} + \hat{z}$

Since if the ray makes a small angle w.r.t. z-axis, then $\boxed{ds \simeq dz}$.Such a ray is called a paraxial ray.

In virtually all the propagation problems we will consider for the rest of the course, we will be concerned only with paraxial rays.

Q: what is the magnitude of the error we are making in the paraxial approximation?

$$ds^2 = dx^2 + dy^2 + dz^2$$

A: $ds = dz\sqrt{1 + \left(\frac{dx}{ds}\right)^2 + \left(\frac{dy}{ds}\right)^2} \simeq dz\left[1 + \frac{1}{2}\left(\frac{dx}{ds}\right)^2 + \frac{1}{2}\left(\frac{dy}{ds}\right)^2\right]$

⇨ A ray must satisfy $\left|\frac{dx}{dz}\right|, \left|\frac{dy}{dz}\right| \ll 1$

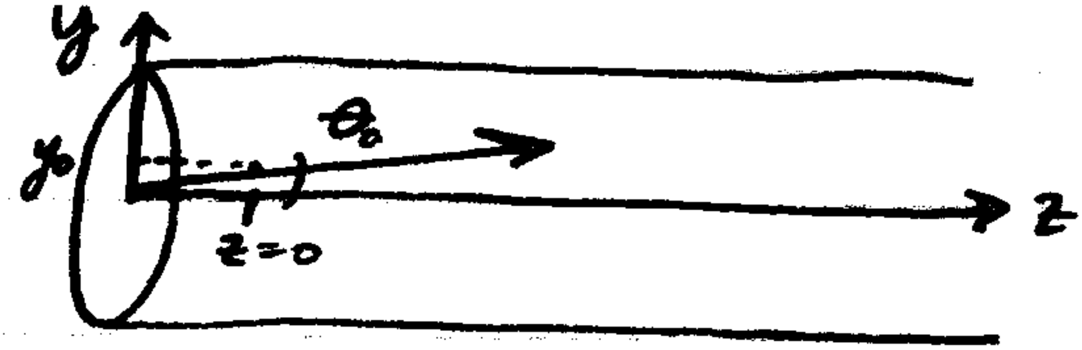
The ray eqn. then has only two components we must consider:

From: $\frac{d}{dz}(n\hat{s}) = \frac{d}{dz}(n\frac{dx}{dz}\hat{x} + n\frac{dy}{dz}\hat{y} + n\hat{z}) = \nabla n$

$$\frac{d}{dz}(n\frac{dx}{dz}) \simeq \frac{\partial n}{\partial x}, \quad \frac{d}{dz}(n\frac{dy}{dz}) \simeq \frac{\partial n}{\partial y}$$

<u>Meridional Rays</u>

Let's first consider a ray which interests the z-axis. We choose the y- axis to be oriented such that the ray is in the y-z plane .



Clearly, by symmetry the z-axis is the "<u>optical axis</u>" of the system .Any ray which propagates such that it crosses the optic axis is known in geometrical optics as a <u>meridional</u> ray .Rays which never interest the optic axis are known as <u>skew rays</u> .

Note that, also by symmetry, the meridional ray will remain in the y-z plane as it propagates. The ray bends only due to $\frac{\partial n}{\partial y}$.

We must therefore solve $\frac{d}{dz}(n\frac{dy}{dz}) = \frac{\partial n}{\partial y}$, subject to the initial conditions

$$y(0)=y_0, \left.\frac{dy}{dz}\right|_0 = \tan\theta_0 \simeq \theta_0 (paraxial)$$

We also need to be given the refractive index profile.

A common form is

$$n^2(y) = n_0^2(1-\alpha^2 y^2)$$

Fiber or slabs of this form are called "Selfoc" elements.

If $\alpha^2 y^2 \ll 1$ for all y of interest (i.e. small – angle paraxial rays), then we can make the approximation

$$n(y) \simeq n_0(1-\frac{1}{2}\alpha^2 y^2) \qquad \text{(parabolic index profile)}$$

$$\frac{\partial n}{\partial y} = -n_0\alpha^2 y$$

Ray equation:

$$\frac{d}{dz}(n\frac{dy}{dz}) = n\frac{d^2y}{dz^2} = -n_0\alpha^2 y$$

$$\frac{d^2y}{dz^2} = -\frac{n_0}{n}\alpha^2 y \simeq -\alpha^2 y$$

to the same degree of accuracy

(write $\dfrac{d^2y}{dz^2} = -\dfrac{n_0\alpha^2 y^2}{n_0(1-t\alpha y^2)}$

r expand dimen + keep only 1^{st} term)

⇨ Solution

$$y(z) = A\cos\alpha z + B\sin\alpha z$$

$$y(0) = y_0 \Rightarrow A = y_0$$

$$\left.\frac{dy}{dz}\right|_0 = \theta_0 \Rightarrow \alpha B = \theta_0 \Rightarrow B = \frac{\theta_0}{\alpha}$$

$$y(z) = y_0\cos\alpha z + \frac{\theta_0}{\alpha}\sin\alpha z$$

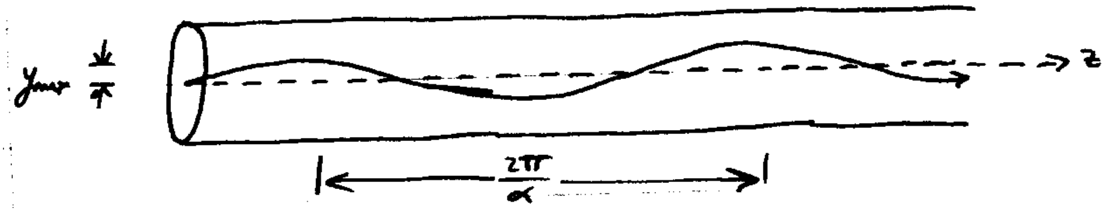
Note that

$$\frac{dy(z)}{dz} = \theta(z) = -y_0\alpha\sin\alpha z + \theta_0\cos\alpha z$$

Thus we see that paraxial meridional rays propagate periodically, crossing the z-axis with a period or "path " $2\pi/\alpha$.

Note that the maximum value y atfmins is

$$y_{\max} = \sqrt{y_0^2 + (\frac{\theta_0}{\alpha})^2}$$



Note that the ray is always bending towards the higher index of refraction, as Snell's law requires. We may plot a whole family of paraxial meridional rays:

(same period for each : only θ_0 and hence $y_{\max}$ varies.)

Skew rays:

Now x can vary as well as y .The index profile is written as

$$n^2 = n_0^{\;2}(1-\alpha^2\rho^2) = n_0^{\;2}\left[1-\alpha^2(x^2+y^2)\right]$$

As before, we assume $\alpha^2(x^2+y^2) \ll 1$, so we have

$$n \simeq n_0\left[1-\alpha^2(x^2+y^2)\right]$$

$$\frac{d^2x}{dz^2} = -\alpha^2 x, \frac{d^2y}{dz^2} = -\alpha^2 y$$

⇨ Both x(z) and y(z) harmonic functions with period $2\pi/\alpha$.

Initial conditions: ray position = (x_0, y_0)

ray angle = $\left(\theta_{x_0}, \theta_{y_0}\right)$

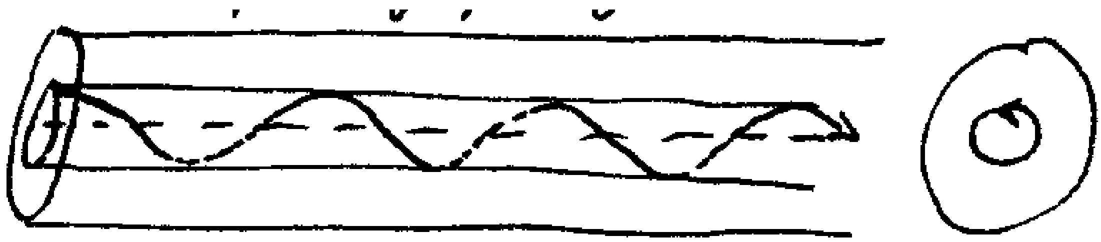
For convenience, we can choose x_0 =0 (ok since choice of origin is arbitrary).Then

$$x(z) = \frac{\theta_{x_0}}{\alpha}\sin\alpha z$$
$$y(z) = y_0\cos\alpha z + \frac{\theta_{y_0}}{\alpha}\sin\alpha z$$

($\theta_{x_0} = 0$ corresponds to meridional case.)

Special case : $\theta_{y_0} = 0, \theta_{x_0} = \alpha y_0$

⇨ $$\left.\begin{aligned} x(z) &= y_0\sin\alpha z \\ y(z) &= y_0\cos\alpha z \end{aligned}\right\}\text{helical ray along} \rho = y_0 = \text{ constant}$$



General case: more complicated helical paths are generated

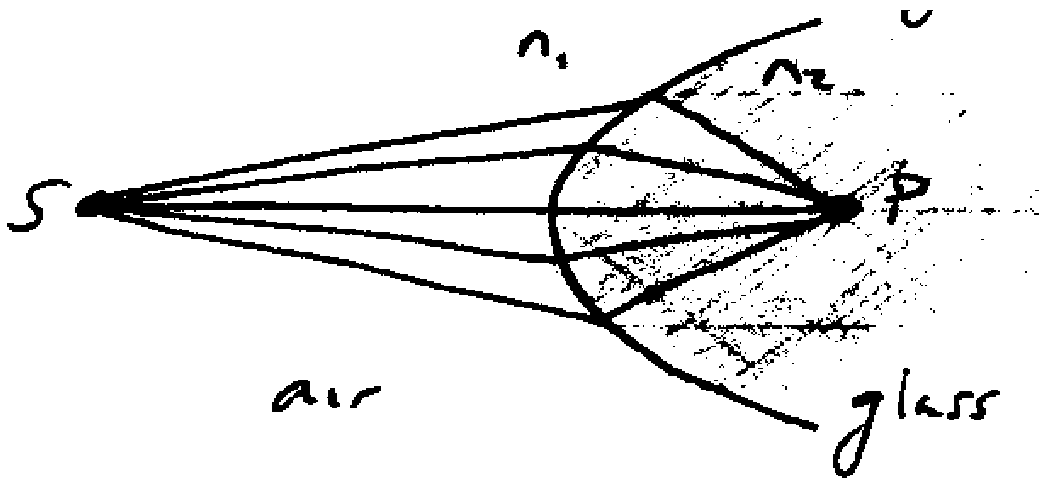
Geometrical Optics of Lenses and Imaging Systems

Our goal in this section is to formulate a general approach to paraxial systems of centered lenses, such as are generally used in most imaging systems and which will be sufficient to describe most laboratory situations. Generalizing to non-paraxial systems and non-cylindrically-symmetrical systems would take us much too far afield, and we will leave that subject to the professional lens designer. The design of such systems is best carried out using computer programs which trace rays through an arbitrary optical system, simply applying Snell's law at each interface.

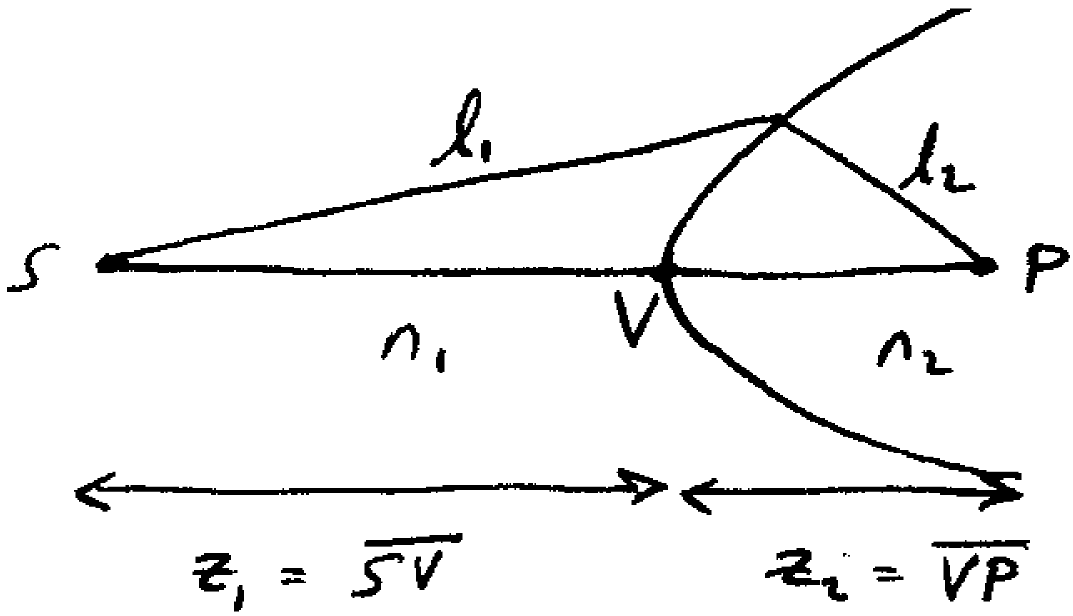
For paraxial, cylindrically symmetrical systems, we can formulate the propagation in terms of ray matrices, which will gives as a very useful analytic technique to describe simple optical systems. We shall see later in the course that there is an intimate connection between these ray matrices and paraxial diffraction theory. As you will see in S39, these ray matrices are also tremendously useful for designing laser cavities ("optical resonators").

Before we embark on our formal treatment of paraxial ray propagation through lens systems , we should digress briefly to remind ourselves of the main ideas and results of "kindergarden" geometrical optics.

Review of Basic Geometrical Optics (See Hecht chap.5 for a review)
Image formation by a refracting surface:
Consider what Young (in Optics+lasers) calls the "len" – just the refraction at a curved air-glass interface.



The point P is an image of the object point S if all the rays (within some solid angle) from S are refracted by the interface so that they go through P.
This happens if and only if the optical path length is the same for all rays, according to Fermat's Principle.

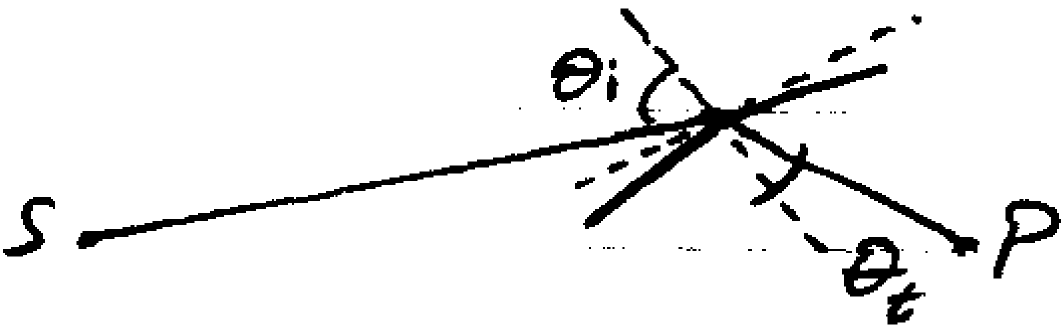


Fermat => $\underline{n_1 \, l_1 + \, n_2 = \text{constant}}$

This is the equation of a "Cartesian oval " of revolution

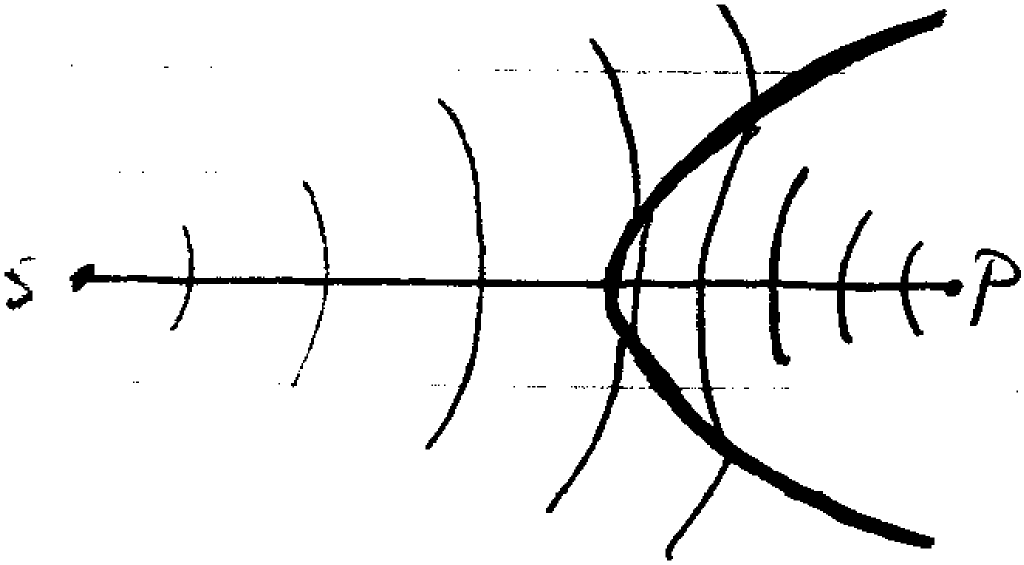
Fermat's Principle defines the shape of the surface required to image S to P. There are two other intuitive pictures you should have, additionally, of how focusing occurs:

(i) Snell 's law of refraction



For the correct surface, all rays refract towards P.

(ii) Following wavefronts:



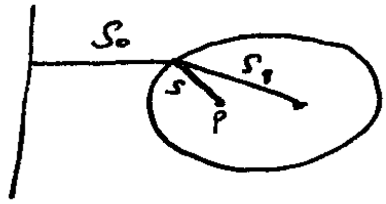
The on-axis wavefronts are delayed with respect to the off-axis ones, which reverses the sign of the curvature

⇨ The wavefronts converge on P.

Special case (e.g. in Hecht P.150): Sat $-\infty$ (z= $-\infty$)

It can be shown that the Cartesian oval in this case is just an ellipse

$S_1 + S =$ constant



Or $\dfrac{S}{S_0}$ = constant (eccentricity)

Practical lenses:

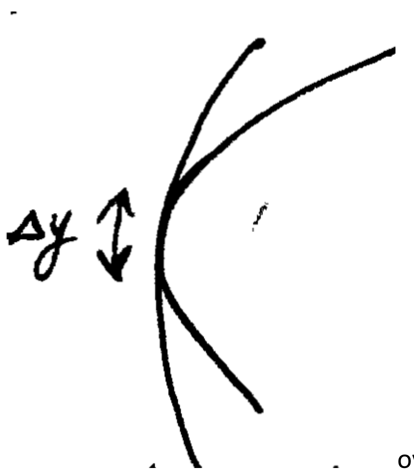
There are two obvious problems with our Cartesian "len."

(i) Not a sphere ."Aspheres" are very difficult and expensive to make , especially with high quality surface with roughness $\ll \lambda$

(ii) The image is inside the glass. We want it somewhere where we can look at it or record it.

Solution: the sphereical lens.

Basic idea: for paraxial ray , which are very close to the optical axis, the Cartesian oval can be approximated by a sphere :



over range Δy, the two surfaces are nearly the same.

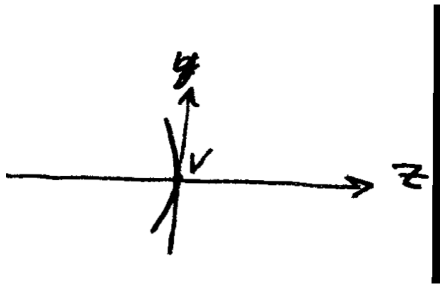
For small angles, the spherical surface will form reasonably sharp images. The imperfections in the image due to the deviations from the correct surface are "spherical aberrations."

plano-convex plano-concave bicoavex meniscus etc.

Of course, the way to calculate the ray paths through such lenses will be to (i) calculate refraction at the first spherical surface, and then (ii) calculate the refraction at the second surface. For thin lenses, the propagation distance in the glass is neglected, but for thick lenses it is not. We shall carry out this procedure next time, utilizing the matrix formalism, but first let's remind ourselves of the basic imaging properties of thin lenses.

Sign convention (Guenther):

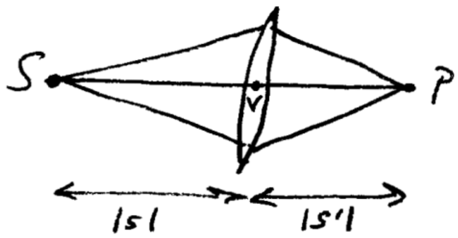
- Optical axis =z axis
- vertex of surface under consideration V is positioned at origin.
- distances to left of V are negative
  to right of V are positive



thin lens equation

object point at z=s <0
image point at z= s'
lens focal length =f

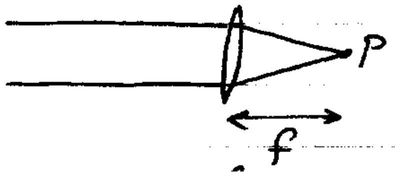
$$\boxed{\frac{1}{S'} - \frac{1}{S} = \frac{1}{f}}$$



Special cases

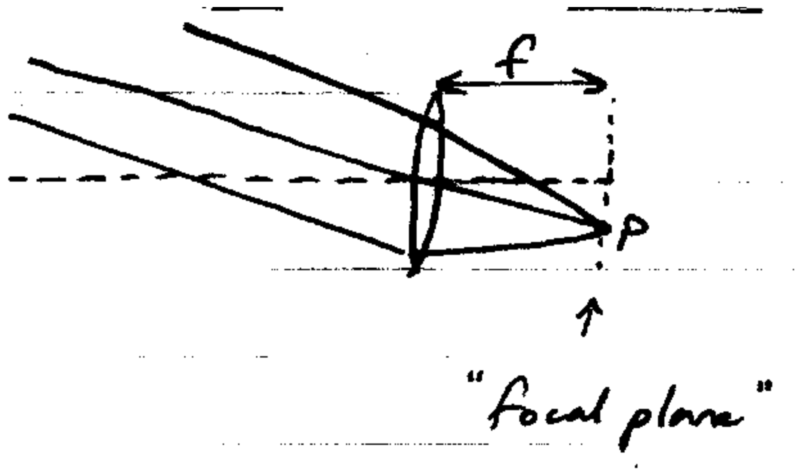
(i) $S = -\infty \Rightarrow S' = f'$

(a) f>0 (positive lens) => rays coverage to focus

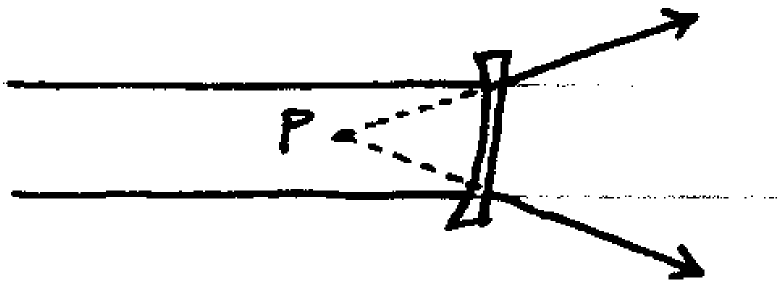


P="real image "

Note procedure for off-axis rays: a ray through the center of a thin lens is undiverted:



(b) $f < 0 \Rightarrow S' = f < 0$

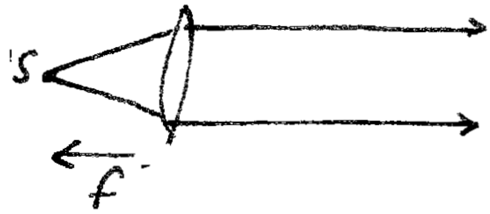


The rays diverge from the negative lens, such that they appear to come from point P .In this case P is called a virtual image.

(The real or virtual test: can you put a screen there and see an image on the screen?)

(ii) $S = -f, f > 0 \Rightarrow S' = \infty$

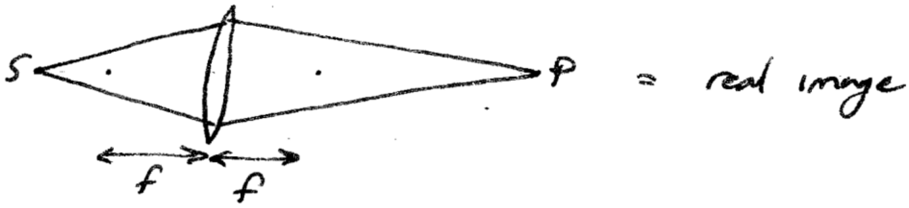
This is just the trivial extension of (i)-(a)



The lens is said to "collimate" the rays from S .

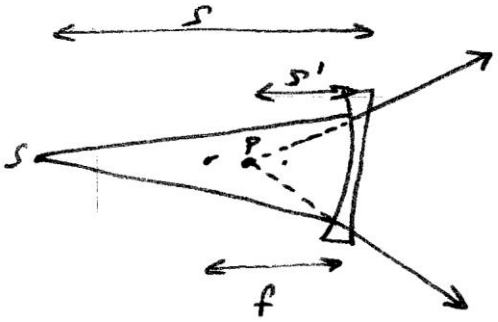
(iii) S<-f

(a) f>0



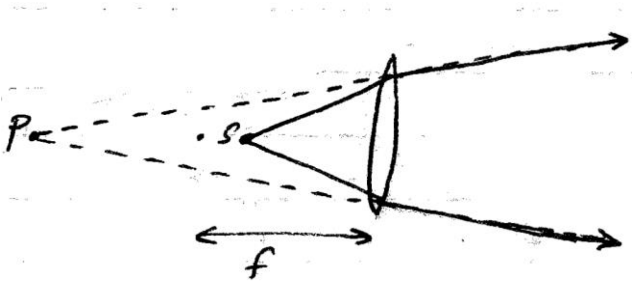
(b) f<0

exercise for reader: $S' < 0$ and $|S'| > \frac{|f|}{2}$



(virtual image of P)

(iv) f>0. –f<S<0 => $S' < 0$ (exercise for reader)



-virtual image
-rays reflect towards axis

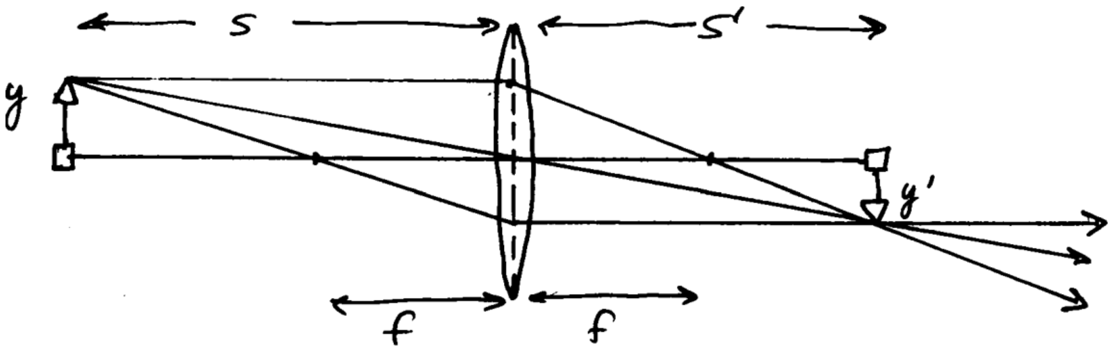
Image formation by ray tracing

The image of an object can be found by tracing just a few strategic rays. All you need to know is

(i) A ray through the center of a thin lens is undeviated.

(ii) Collimated rays pass through the focus (+ vice versa).

Ex. single positive thin lens



Magnification = $\frac{y'}{y} < 0$ => image is <u>inverted</u> and <u>real</u>

Note that I used three rays, when in fact only two are necessary.

Ex. Single negative lens

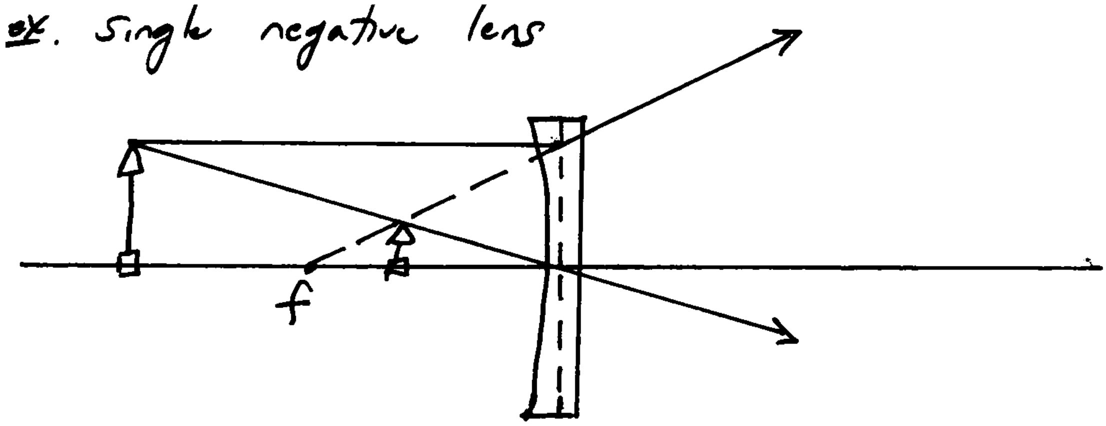


Image is upright, virtual, and when $|S| > |f|$, smaller than the object.