Lecture 21

Geometrical Optics

For many applications in optics, it is unnecessary to carry the full weight of Maxwell's equations or even scalar wave equations to describe an optical system adequately. As you are well aware, it is frequently sufficient to describe the propagation of light by means of rays.

In this section of the course, it is our goal to give a concise treatment of those areas of geometrical optics which you are most likely to use in the future, or which should be part of the basic tool box of any optical scientist. Geometrical optics is itself a huge field, but the essentials required in everyday physical research are quite straightforward. I will assume a certain familiarity with geometrical optics at the "sophomore physics" level.

Our first task will be to move from wave equations to ray equations, and to understand the conditions under which geometrical optics is to adequate. We shall also, show how ray equations can be obtained from Fermat's Principle. Following this, we will be in a position to study light propagation in inhomogeneous media from a ray point of view. We will then discuss paraxial rays and optical imaging, with some discussion of optical instruments. Some treatment of aberrations will also be given.

From Waves to Ray Optics

We have seen that the propagation of time–harmonic light waves is described by the Helmholtz equation

$$\nabla^2\vec{E} + k^2\vec{E} = 0$$

Where $k^2 = \dfrac{n^2\omega^2}{c^2}$

In many systems, the index of refraction varies with position: $n \to n(\vec{r})$ .Example include the atmosphere, gradient –index (GRIN) optics, and "thermal lensing" in crystals.

Define the free-space wavevector as $k_0 = \dfrac{\omega}{c}$.

We know the plane-wave solution to the Helmholtz eqn, based on its form, let's try a solution which looks similar:

(*) $$E(\vec{r}) = A(\vec{r})e^{-ik_0 S(\vec{r})}$$

(1) It's a scalar wave, so polarization is ignored (usually the case in geometrical optics)
(2) The amplitude and phase A and S are functions which will be assumed to vary slowly with position.
(3) Notation: S is called the "eikonal" function. The form of the solution (*) is often referred to as a "WKB" solution (for Wentzel, Krammers, and Brillouin, who developed it in the contest of quantum mechanical wave equations). Definitions of S vary in different treatments of WKB and geometrical optics, so one must be careful.

A common situation is when the index of refraction varies along only one spatial dimension, e.g. n=n(z).

Then our trial solution is $E(z,t) = A(z)e^{i[\omega t - k_0 S(z)]}$

Or $E(z) = A(z)e^{-ik_0 S(z)}$

To substitute into the Helmholtz eqn. we need

$$\frac{dE}{dz} = \frac{dA}{dz}e^{-ik_0 S} - ik_0 \frac{dS}{dz} A e^{-ik_0 S}$$
$$\frac{d^2E}{d^2z} = [\frac{d^2A}{d^2z} - 2ik_0 \frac{dS}{dz}\frac{dA}{dz} - ik_0 \frac{d^2S}{d^2z} A - A\left(k_0 \frac{dS}{dz}\right)^2]e^{-ik_0 S}$$

Substituting to the Helmholtz eqn. and dividing out $e^{-ik_0 S}$:

$$[\frac{d^2A}{d^2z} - A\left(k_0 \frac{dS}{dz}\right)^2] - i[2k_0 \frac{dS}{dz}\frac{dA}{dz} + Ak_0 \frac{d^2S}{d^2z}] + k^2 A = 0$$

The real and imaginary parts must separately vanish

$$\frac{d^2A}{d^2z} + [\frac{n(z)^2\omega^2}{c^2} - \left(k_0 \frac{dS}{dz}\right)^2]A = 0$$

And $$2k_0 \frac{dS}{dz}\frac{dA}{dz} + Ak_0 \frac{d^2S}{d^2z} = 0$$

Now if the index of refraction n(z) varies “slowly” with position (on the scale of a wavelength), then A and S should be slowly varying functions, in which case $\frac{d^2A}{d^2z}$ should be negligible .

WKB approximation: neglect $\frac{d^2A}{d^2z}$, so from the real part of the Helmholtz eqn.

$$\frac{n(z)^2\omega^2}{c^2} - k_0 \left(\frac{dS}{dz}\right)^2 = 0 \qquad k_0 = \omega/c$$

$$\left(\frac{dS}{dz}\right)^2 = n^2(z)$$

$$\frac{dS}{dz} = n(z)$$

Or $$\boxed{S(z) = \int_0^z n(z')dz'}$$ WKB solution for the phase

Thus $$E(z) = A(z)e^{-ik_0 \int_0^z n(z')dz'}$$

$A(z)$: Slowly varying amplitude function (to be determined)

$\int_0^z n(z')dz'$: WKB solution looks locally (i.e. on a small region of space) like a plane wave

i.e. the solution is a plane wave whose wavevector $k_0 S$ varies smoothly and slowly with position

Physical meaning of S:

Suppose n = constant

$$n\int_0^z dz' = nz$$

Index $\times$ physical length $\equiv$ "optical path length "

So even in the case where n is spatially varying, then the optical path length is $S = \int_0^z n(z')dz'$

Note that the phase

$$\phi = k_0 S = \frac{2\pi}{\lambda} S = 2\pi \frac{\text{optical path length}}{\text{wavelength}}$$

So S/λ is the "number of waves "in a given O.P.L.S.

Ex. n=1 ,S waves

n=2, S waves

n=1+az, S waves

(Although in the latter figure, we are probably in a regime where WKB is not valid, since the index is not varying very slowly on the scale of λ .

In general n can vary in all three spatial dimensions $n = n(\vec{r})$ .In this case the trial solution is

$$E(\vec{r}) = A(\vec{r})e^{-ik_0 S(\vec{r})}$$

Substituting into the Helmholtz eqn. and carrying out similar algebra to our 1-D case yields

$$\nabla^2 A + [n^2 - (\nabla S)\cdot(\nabla S)]k_0^2 A = 0$$

And $2(\nabla S)\cdot(\nabla A)+\left(\nabla^2 S\right)A=0$

Note the similar form to the 1-D case. As before, if the index varies slowly as a function of position, we neglect $\nabla^2 A$ (WKB approximation)

The first equation then gives

$$\nabla S\cdot\nabla S=\boxed{|\nabla S|^2=n^2(\vec{r})}$$

This is known in the optics literature as the eikonal equation.

Note that when $S(\vec{r})=$ constant, this describes a surface of constant phase (the phase is $\phi=k_0S$ everywhere on that surface)

Consider some special cases:

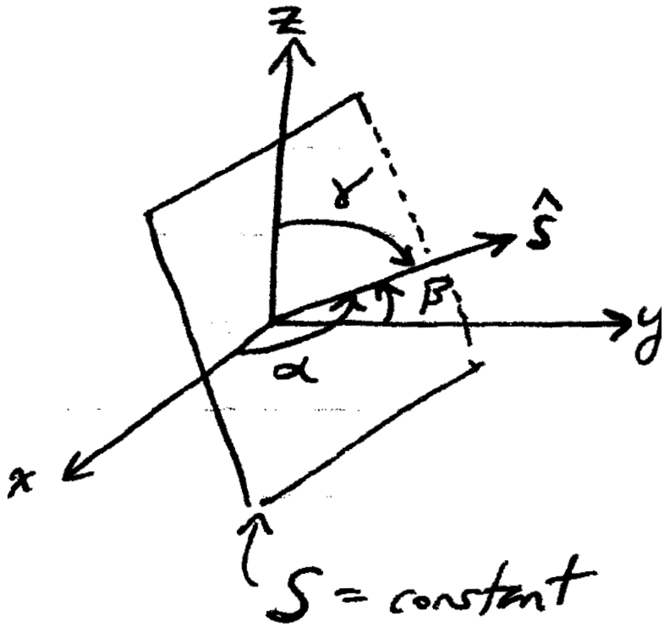
(i) $n(\vec{r})$ = n = constant (uniform medium)

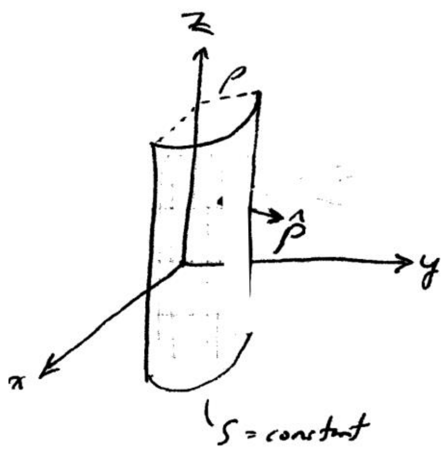
Suppose initial solution is a plane wave, whose initial wave surface has a normal with direction cosines α,β,γ :

Then $S=n(\alpha x+\beta y+\gamma z)$

And $\nabla S=n\hat{S}$

Where $\hat{S}=\alpha\hat{x}+\beta\hat{y}+\gamma\hat{z}$





(ii) n=constant; initial wave surface =cylinder

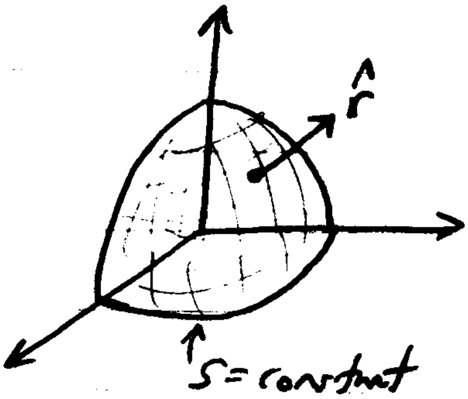
⇨ $S=n\rho$ $\hat{s}=\hat{\rho}$

$\nabla S=n\hat{\rho}\rightarrow\nabla S=n\hat{s}$

Where ρ = radial coordinate

(iii) n=constant, initial wave =sphere

$$S = nr$$
$$\nabla S = n\hat{r}$$
$$\Rightarrow \quad \nabla S = n\hat{s}$$



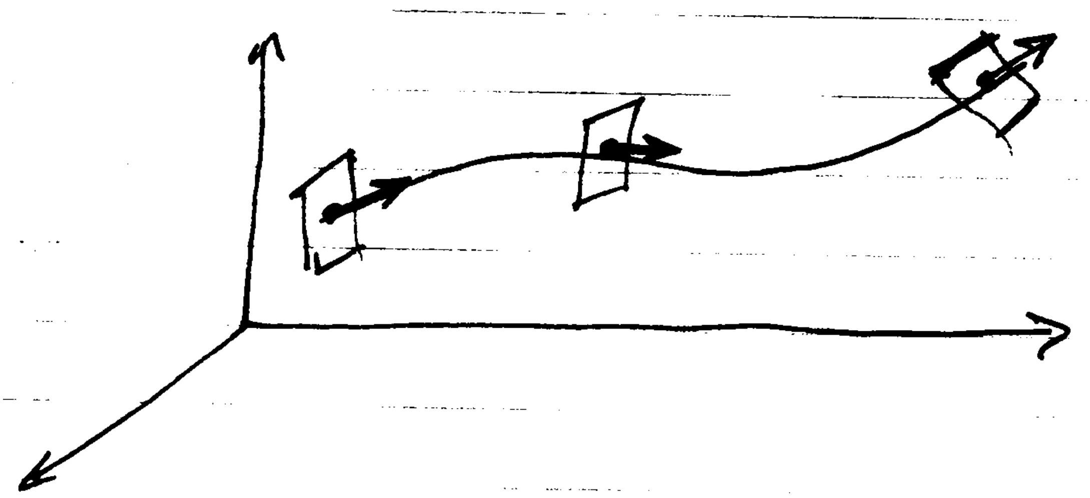
In general, we have

$$\boxed{\nabla S = n(\vec{r})\,\hat{s}(\vec{r})}$$

Where $\hat{s}(\hat{r})$ is a unit vector normal to the surface at the position $\hat{r}$.

Recall ∇S = grad S is a vector perpendicular to the surface S =constant.

Def. a ray is the path taken by $\hat{s}$ as the wave propagates through space



It is this physical picture which establishes the regime of geometrical optics. If in some local region the wave acts as a plane wave, then we can determine the normal to the wave surface and the path of normal defines a ray.

When is this picture of "local plane waves" valid?

Precisely when $\nabla^2 A$ is negligible, i.e. A varies slowly on the scale of a wavelength.

This leads us to a popular description of geometrical optics: it is the limiting case when the wavelength tends to zero. (Clearly if we send the wavelength to zero, $\nabla^2 A$ is always negligible, and the propagation will be defined entirely by rays.)

What is the difference in perspective between geometrical optics and the WKB picture (especially given the similarity of the mathematics)??

Geometrical optics: description of propagation entirely in terms of the rays *

WKB/ wave optics: description in terms of the wave surfaces and amplitudes

*(in the limit $\lambda \to 0$, the wave surfaces become meaningless; we are left only with the path of surface normal)

We have one more thing to learn about the eikonal S:

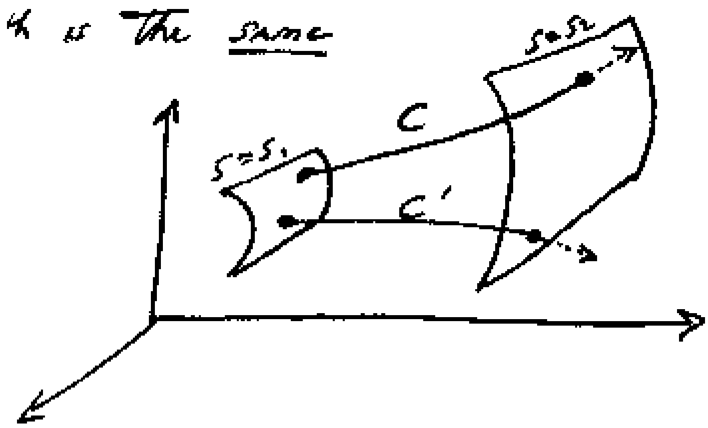
Consider $|\nabla S| = n$, and integrate between two positions A and B :

$$\int_A^B |\nabla S|\, dS = S(\vec{r}_B) - S(\vec{r}_A) = \underbrace{\int_A^B n dS}$$

Optical path length from A to B

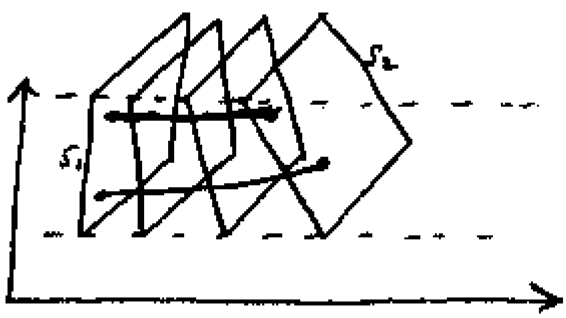
Thus the difference $S(\vec{r}_B) - S(\vec{r}_A)$ represents the optical path length between the two points.

Note that, if you follow a ray from one surface $S = \text{constant} \equiv S_1$, to another surface $S = \text{constant} \equiv S_2$, the optical path length is the same



$$\int_C ndS = \int_{C'} ndS = S_2 - S_1$$

A more interesting case: (bad picture: rays should curve more!)



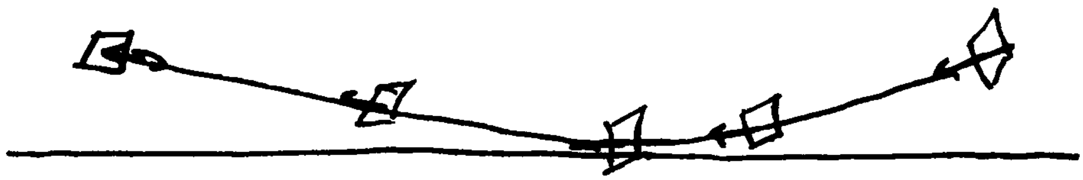
Example from atmosphere optics: the mirage

Normally the index of refraction decreases continuously with height in the atmosphere, since the density decreases.
When the sun heats the surface (e.g. of a road or sand), the air just next to the surface is hot and thus has a lower index than the air above it .

$$n = 1.000291 - (1\times10^{-6})\,T$$

The experimental result:

From a WKB wave point of view, the local plane waves change direction because the index seen below the ray path is lower than above :



Note that the mirage is the result of total internal reflection! (the only difference from the familiar case being that the ray path is smooth +slowly varying .)