0.1 Pulse Manipulation + Space-Time Analogies

We now have the tools to control the dispersion of optical pulses. We shall see a number of applications as we go, but the most obvious are (1) to compensate for material dispersion, and (2) to stretch pulses to keep their peak power low on either propagation or amplification, and then to recompress the pulse. The manipulation of electromagnetic pulses was first done extensively in the 40's and 50's in the microwave region of the spectrum with the development of 'chirp radar'. The development of optical technologies in the past 20 years have enabled many similar technologies of pulse manipulation in the visible/near IR spectrum. Our next goal is to provide a physical and very general description of dispersive pulse propagation.

Typical system:

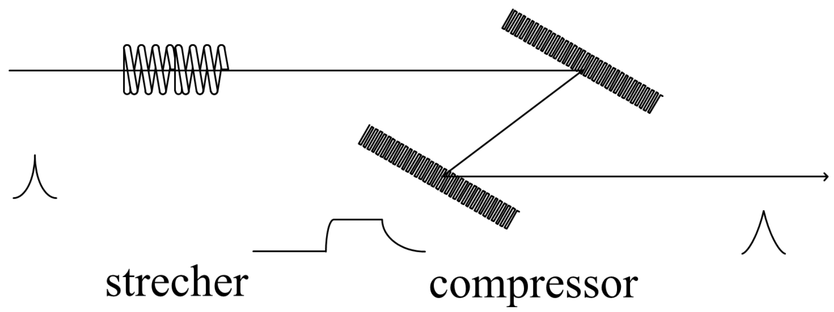


Figure 1: dispersive pulse propagation system.

Perfect recompression:

$$\varphi_s^{''} + \varphi_c^{''} = 0$$

$$\varphi_s^{'''} + \varphi_c^{'''} = 0$$

$$\ldots$$

Let us assume that all phase terms of 3rd order and higher can be neglected

$$\varphi \simeq \varphi(\omega_0) + \varphi^{'}(\omega - \omega_0) + \frac{1}{2}\varphi^{''}(\omega - \omega_0)^2$$

i.e. we need consider **only quadratic phase terms** (sometimes this is called ‘first order optics’)

Let us start by recalling some results for linearly chirped Gaussian pulse propagation

$$\varepsilon(z,t) = e^{i(\omega_0 t - \beta_0 z)} e^{-\Gamma(z)(t-\beta' z)^2}$$

where

$$\Gamma = a(z) - ib(z)$$

$$\frac{1}{\Gamma(z)} = \frac{1}{\Gamma_0} + 2i\beta'' z$$

$$(\tau_p = \sqrt{\frac{2\ln 2}{a}})$$

for b_0=0 (initially transform-limited pulse)

$$\tau_p = \tau_{p0}\sqrt{1 + 4a_0^2\beta''^2 z^2}$$

$$= \tau_{p0}\sqrt{1 + \frac{z^2}{L_D^2}}$$

where L_D =characristic length (doubles pulse in **intensity** in distance $z = z_0$) $= \frac{1}{2a_0\beta''}$

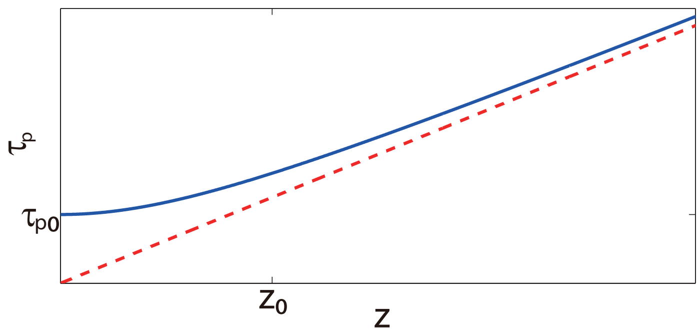


Figure 2: $\tau_p - z$ plot.

This functional form of the Gaussian pulsewidth vs. z should remind you of the **Gaussian beam** in spatial propagation. (For a review of Gaussian beams, see Siegman chap. 17)

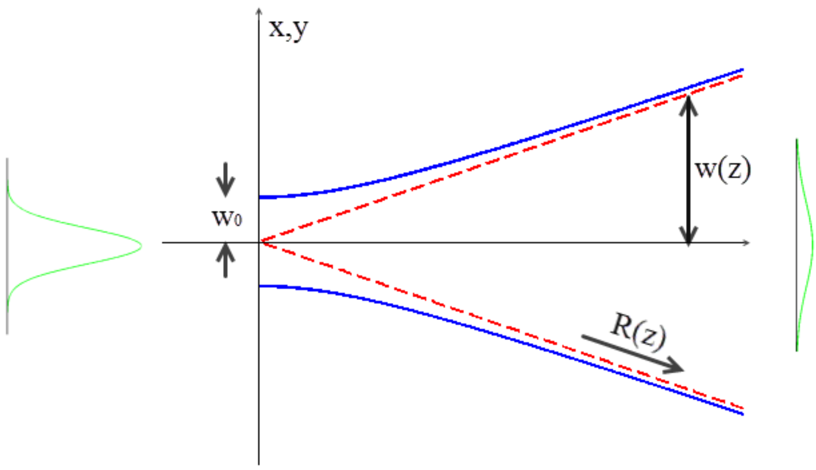


Figure 3: Gaussian pulsewidth vs. z

$w(z)$= beam radius

w_0= 'waist' = minimum radius

$R(z)$ = radius of curvature of phase fronts

In fact, the analogy between the spatial and temporal Gaussians is quite strong. Before going on to establish the general analogy between dispersive pulse propagation and diffractive spatial propagation, we will show the analogous parameters for Gaussian beams + pulses.

Normalized field amplitude of Gaussian beam with waist at z=0:

$$u(x,y,z) = \sqrt{\frac{2}{\pi}} \frac{e^{-i\beta_0 z + i\Psi(z)}}{w(z)} e^{[-\frac{x^2+y^2}{w^2(z)} - i\beta_0 \frac{x^2+y^2}{2R(z)}]}$$

where $w(z) = w_0\sqrt{1+(\frac{z}{z_R})^2}$ = beam radius

$R(z) = z + \frac{z_R^2}{z}$ = wavefront radius

$\Psi(z) = \arctan(\frac{z}{z_R})$, describes π phase shift through focus

z_R is konwn as the Rayleigh length (and describes how fast the beam diverges). It is related to the waist size by

$$z_R = \frac{\pi w_0^2}{\lambda}$$

Thus a smaller waist radius will give a shorter Rayleigh range, corresponding to a larger beam divergence, i.e. larger diffraction angle.

By comparing the functional forms of the spatial and temporal Gaussians, we can draw strong analogies between the two.

Table 1: analogies between spatial and temporal Gaussians

spatial	temporal
waist w_0 $\sqrt{1+(\frac{z}{z_R})^2}$	pulse width τ_{p0} $\sqrt{1+(\frac{z}{l_D})^2}$
Rayleigh range $z_R = \frac{\pi w_0^2}{\lambda}$	characteristic dispersion length $l_D = \frac{1}{2a_0\beta''} \propto \tau_{p0}^2$
wavefront radius $R^{-1}(z)$	chirp parameter $b(z)$

To see the latter relationship, we rewrite R as

$$R(z) = z + \frac{z_R^2}{z} = \frac{z_R^2 + z^2}{z}$$

$$\frac{1}{R} = \frac{z}{z_R^2+z^2} = \frac{z/z_R}{1+(z/z_R)^2}\frac{1}{z_R} \propto \frac{z/z_R}{1+(z/z_R)^2}\frac{1}{\omega_0^2}$$

Now recall we derived the chirp parameter b to be (when b_0=0, no initial chirp):

$$b(z) = \frac{2a_0^2\beta'' z}{1 + (2a_0\beta'' z)^2}$$

with our above definition of l_D, we have

$$b(z) = \frac{z/l_D}{1 + (z/l_D)^2} \cdot a_0 \propto \frac{z/l_D}{1 + (z/l_D)^2}\frac{1}{\tau_{p0}^2}$$

Thus our pricture of pulse propagation in terms of its analogue in Gaussian beam propagation is:

1. at z=0, start with **transform-limited pulse,** which corresponds to a **beam waist**.

2. at z=0, chirp parameter b=0 $\longleftrightarrow$ R=∞

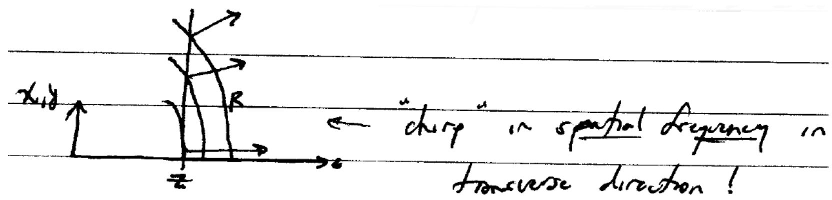
3. as z increases from zero, the pulse spreads in time, corresponding to spatial diffraction. The shorter the pulse (the smaller the beam waist), the more rapidly the pulse spreads (the faster the diffraction).

4. The width of the pulse (in intensity) doubles in a distance l_D; its beam radius doubles in z_R.

5. as $z \longrightarrow z \gg l_0, z_R$

$$R(z) = z,\ b \propto \frac{1}{z}$$

(fixed bandwidth $\Rightarrow$ when the pulse is **very** long, the instantaneous rate of phase change is slow)



recall definition of spatial frequency

$$\omega_{x,y} = -\tfrac{2\pi}{\lambda}\theta_{x,y} \ (\ \theta_{x,y} = \text{angle w.r.t. } z \text{ axis in } x - z \text{ or } y - z \text{ plane})$$

of course, all spatial frequencies are present in Gaussian beam at z=0, but they spread out with propagation ⇒ 'chirp' in spatial frequency.

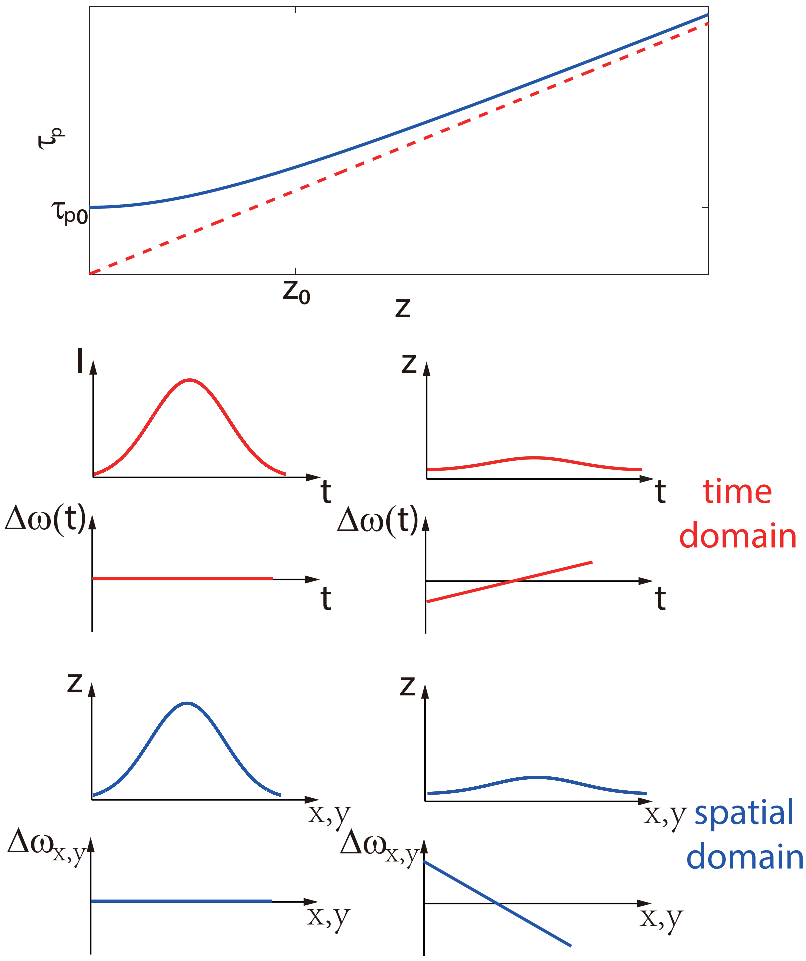


Figure 4: pulse propagation.

Fourier synthesis of a Gaussian beam

At the beam **waist**, all $\vec{k}$ vectors (i.e. all spatial frequency components) are **in phase** at $x = y = 0$ ⇒ 'transform limited' beam.

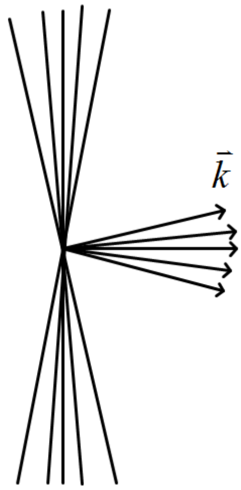


Figure 5: k_x and k_y components cancel, so net $\vec{k}$ is only in z direction $\Rightarrow$ $R(0) = \infty$ (planar wavefront)

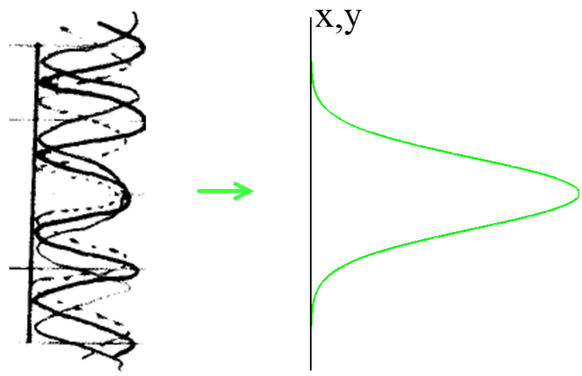


Figure 6: summing an infinite number of plane waves with Gaussian weighting produces a Gaussian beam with minimum width (w_0)

As the beam propagates, each spatial frequency component propagates at an angle $\theta_{x,y}$, and these plane waves are no longer all in phase at $x = y = 0$. When $z \gg z_R$, the positions of 'stationary phase', where spatial frequency 'groups' are in phase, correspond to spherical wavefronts with radius $R(z)$:

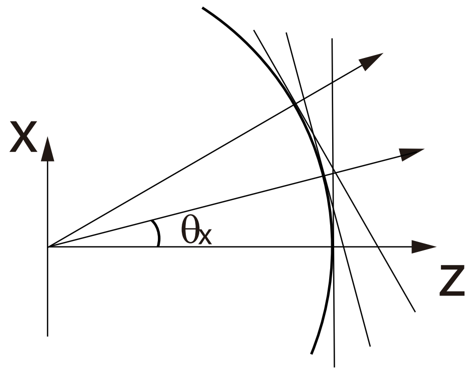


Figure 7: 'chirp corresponds to spatial frequency components $\omega_x < 0$ occuring at $x > 0$. (diffraction↔negative dispersion)

The analogy between dispersive pulse propagation and spatial diffraction actually goes well beyond just Gaussian beams. Examining the analogy in its most **general** form will in fact lead us to a deeper understanding of pulse compression, and enable invention of a few novel devices as well.

Parabolic Eqn. describing dispersive pulse propagation:

$$\frac{\partial \tilde{E}}{\partial z} + \frac{1}{v_g}\frac{\partial \tilde{E}}{\partial t} = \frac{i}{2}\beta'' \frac{\partial^2 \tilde{E}(z,t)}{\partial t^2}$$

where $\tilde{E}(z,t)$ is the pulse envelope

$$\tilde{\varepsilon}(z,t) = \tilde{E}(z,t)e^{2[\omega_0 t - \beta(\omega_0)z]}$$

For convenience, we can make a change of variables to a coordinate system travelling at the group velocity of the pulse:

$$\tau = t - \frac{z}{v_g};\ \xi = z$$

⇒ parabolic eqn. is

$$\frac{\partial \tilde{E}(\xi,\tau)}{\partial \xi} = \frac{i}{2}\beta'' \frac{\partial^2 \tilde{E}(\xi,\tau)}{\partial \tau^2}$$

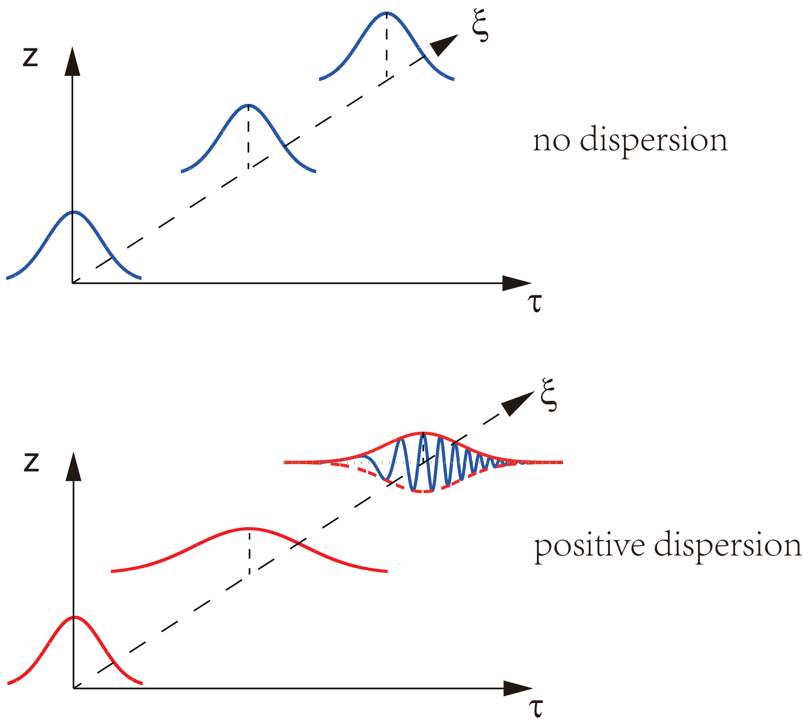


Figure 8: dispersive/nondispersive pulse propagation.

- peak moves at $v_g(\omega_0)$
- ξ is the 'local' position corrdinate (in reference from moving at v_g)

0.2 Paraxial Diffraction

(for review, see Siegman chap 16)

We will follow the treatment (and for the most part the notation as well) of Kolner in *JQE* **30,** 1951 (1994).

As usual, start with the wave eqn.

$$\nabla^2 \vec{E} = \mu\epsilon \frac{\partial^2 \vec{E}}{\partial t^2}$$

where the dielectric material response is in in ϵ. Assume a monochromatic scalar wave

$$\varepsilon(x,y,z,\omega) = E^{'}(x,y,z)\delta(\omega-\omega_0)$$

$\Rightarrow$ Helmholtz eqn.

$$(\nabla^2 + k^2)E^{'}(x,y,z) = 0,\ k^2 = \mu\epsilon\omega_0^2$$

Consider propagation along z-axis, so that the normals to the phase fronts (i.e. the rays) are confined nearly along the z-axis. These are called paraxial rays:

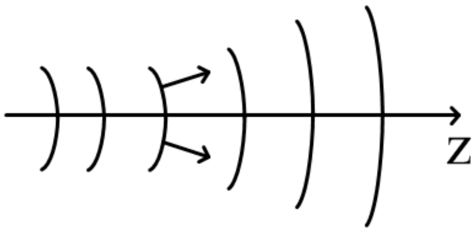


Figure 9: paraxial rays.

$$E^{'}(x,y,z) = E(x,y,z)e^{-ikz}$$

$E(x,y,z)$ is slowly varying (in space) envelope function, e^{-ikz} is the rapidly varying phase.

Consider

$$\frac{\partial^2 E^{'}}{\partial z^2} = \frac{\partial}{\partial z}[\frac{\partial E}{\partial z}e^{-ikz} - ikEe^{-ikz}]$$

$$= (\frac{\partial^2 E}{\partial z^2} - k^2 E - 2ik\frac{\partial E}{\partial z})e^{-ikz}$$

$\Rightarrow$ Helmholtz eqn.

$$\frac{\partial^2 E}{\partial x^2} + \frac{\partial^2 E}{\partial y^2} + \frac{\partial^2 E}{\partial z^2} - 2ik\frac{\partial E}{\partial z} = 0 \quad (dividing\, out\, e^{-ikz})$$

Paraxial Approximation:

$$|\tfrac{\partial^2 E}{\partial z^2}| \ll |\text{ all other terms in Helmholtz eqn.}|$$

Physically, this means that the change in the field envelope with propagation is **slow**, both with respect to a **wavelength,** and with respect to the scale of the transverse profile.

$\Rightarrow$ paraxial wave eqn.

$$\frac{\partial E}{\partial z} = -\frac{i}{2k}\nabla_T^2 E$$

Comparing this to the parabolic eqn. mentioned above, we see a strong resemblance (in fact, if we consider only one transverse dimension in the paraxial eqn., then they are mathematically identical).

Thus we can translate between the two pictures by making the identifications

Table 2: spatial propagation versus dispersive pulse propagation

spatial propagation	dispersive pulse propagation
z	ξ
x	τ
$\frac{1}{k}$	$-\beta''$

Note that $\frac{1}{k} \sim \lambda$ measures the rate at which a beam will expand by diffraction (for a given aperture size, the long wavelengths differact more rapidly than the short wavelengths). Similarly β'' measures the rate at which a short pulse expands in a dispersive medium (higher GDD means the pulse stretches more rapidly).

note: positive GVD is $\beta'' > 0$

$\Rightarrow$ in some sense diffraction is from space corresponds to 'negative dispersion'

The problem, of course, is to **solve** the propagation problem; i.e. **given** a pulse envelope $\tilde{E}(0, \tau)$, what is the envelope $\tilde{E}(\xi, \tau)$ after propagating a distance ξ in a dispersive medium? One practical way would be to just crank through a numerical solution to the parabolic eqn. directly (i.e. a finite-difference solution). However, we can also follow the diffraction-propagation analogy and obtain useful integral solutions.

0.3 paraxial wave eqn. solution

Given input at $z = 0$, $E(x, y, 0)$, we know that a spherical wave

$$E' = \frac{e^{-ikr}}{r}$$

is an exact solution to the Helmholtz eqn. Now consider this wave along the z-axis

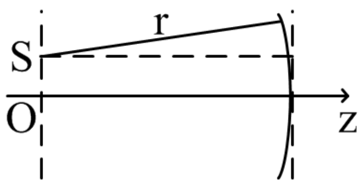


Figure 10: a spherical wave along the z-axis

$r=$ distance from source point to observation point

$$= \sqrt{(x - x_0)^2 + (y - y_0)^2 + z^2}$$

$$= z\sqrt{1 + \frac{(x - x_0)^2}{z^2} + \frac{(y - y_0)^2}{z^2}}$$

Fresrel approximation

for observation points close to the z-axis (i.e. for **paraxial rays**)

$$r \simeq z[1 + \frac{(x - x_0)^2}{2z^2} + \frac{(y - y_0)^2}{2z^2}]$$

$$= z + \frac{(x - x_0)^2}{2z} + \frac{(y - y_0)^2}{2z}$$

$\Rightarrow$ spherical wave is approximately

$$E' \simeq \frac{e^{-ikz}}{z} e^{-ik[\frac{(x-x_0)^2}{2z} + \frac{(y-y_0)^2}{2z}]}$$

As before, writing

$$E' = E(x,y,z)e^{-ikz}$$

gives for the envelope

$$E(x,y,z) = \frac{1}{z} e^{-ik[\frac{(x-x_0)^2}{2z} + \frac{(y-y_0)^2}{2z}]}$$

(the 'paraxial-spherical wave')

It is easy to verify that this is an **exact solution** to the **paraxial wave equation.**

In other words, the approximations leading to the paraxial wave eqn. are exactly those leading to the paraxial-spherical wave (i.e. terminating the phase term so it is **quadratic** in the transverse spatial variables x, y is equavalent to neglecting $\frac{\partial^2 E}{\partial z^2}$).

check solution:

$$\frac{\partial E}{\partial x} = -\frac{ik}{z} E(x - x_0)$$

$$\frac{\partial^2 E}{\partial x^2} = -\frac{ik}{z} E - \frac{ik}{z}(x - x_0)\frac{\partial E}{\partial x}$$

$$= -\frac{ik}{z} E - \frac{ik}{z}(x - x_0)[-\frac{ik}{z} E(x - x_0)]$$

$$= -\frac{ik}{z} E - \frac{k^2}{z^2}(x - x_0)^2 E$$

$$\frac{\partial^2 E}{\partial y^2} = -\frac{ik}{z}E - \frac{k^2}{z^2}(y-y_0)^2 E$$

$$\frac{\partial E}{\partial z} = -\frac{E}{z} - E\{-ik[\frac{(x-x_0)^2}{2z^2} + \frac{(y-y_0)^2}{2z^2}]\}$$

$$\frac{\partial E}{\partial z}\nabla_T^2 E - 2ik\frac{\partial E}{\partial z} = \frac{2ik}{z}E + \frac{k^2}{z^2}[(x-x_0)^2+(y-y_0)^2]E - \frac{2ik}{z}E - \frac{k^2}{z^2}[(x-x_0)^2+(y-y_0)^2]E = 0$$

$\Rightarrow E$ is a solution to the paraxial wave eqn.

Now we can apply the Huygens-Fresnel Principle's: given $E(x_0, y_0, 0)$ in the source plane $z = 0$, consider each point on the wavefront at $z = 0$ to be a **source** of paraxial-spherical waves. The total field at a point (x,y,z) is the **sum** of all these waves.

$$E(x,y,z) = \frac{i}{\lambda z}\int\int E(x_0,y_0,0)e^{-ik[(x-x_0)^2+(y-y_0)^2]/2z}dx_0dy_0$$

(Huygens-Fresnel integral; the $\frac{i}{\lambda}$ comes from a more rigorous approach to the theory)

In one spatial dimension, one would have a slightly different normalization:

$$E(x,z) \simeq \sqrt{\frac{i}{\lambda z}}\int E_0(x_0,0)e^{-ik(x-x_0)^2/2z}dx_0$$

Now by making the variable identifications above, we have the solution to the parabolic eqn. for pulse propagation:

$$E(\tau,\xi) \simeq \frac{1}{\sqrt{\xi}}\int E_0(\tau_0,0)e^{-i(\tau-\tau_0)^2/2\beta''\xi}d\tau_0$$

Thus we can find the complex field after propagation over an arbitrary distance ξ.

It is interesting to consider what happens after propagation over a long distance:

$$\frac{(\tau - \tau_0)^2}{2\beta''\xi} = \frac{\tau^2 - 2\tau_0\tau + \tau_0^2}{2\beta''\xi}$$

i.e. consider $\frac{\tau_0^2}{2\beta''\xi} \ll 1$ or $\xi \gg \frac{\tau_0^2}{2\beta''}$ (This is equivalent to taking the **Fraunhofer limit** in diffraction theory.) i.e. $z \gg \frac{\pi\omega_0^2}{\lambda}$ ($\beta'' \leftrightarrow \frac{1}{b} = \frac{\lambda}{2\pi}$)

$$E(\tau, \xi) \simeq \frac{e^{-i\tau^2/2\beta''\xi}}{\sqrt{\xi}} \int E_0(\tau_0, 0) e^{i\tau\tau_0/\beta''\xi} d\tau_0$$

$\frac{e^{-i\tau^2/2\beta''\xi}}{\sqrt{\xi}}$ is overall quadratic phase factor (on the local time). $\int E_0(\tau_0, 0)e^{i\tau\tau_0/\beta''\xi}d\tau_0$ is Fourier transform of the input pulse.

Example: linearly chirped square pulse = input field $\Rightarrow$ output looks like sinc function.

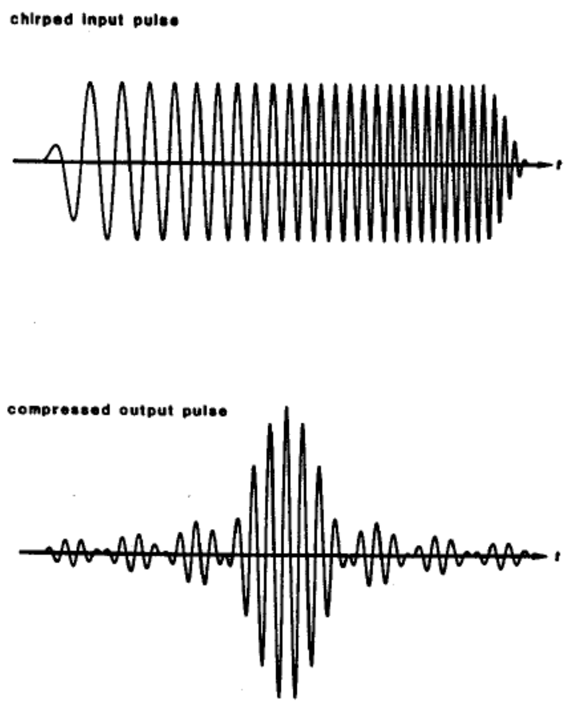


Figure 11: pulse compression and reshaping with a square input pulse envelope.

Recall that exactly the same thing happens in Fraunhofer diffraction; the far field is the Fourier transform of the source field, but with a curved (paraboloidal) phase front.