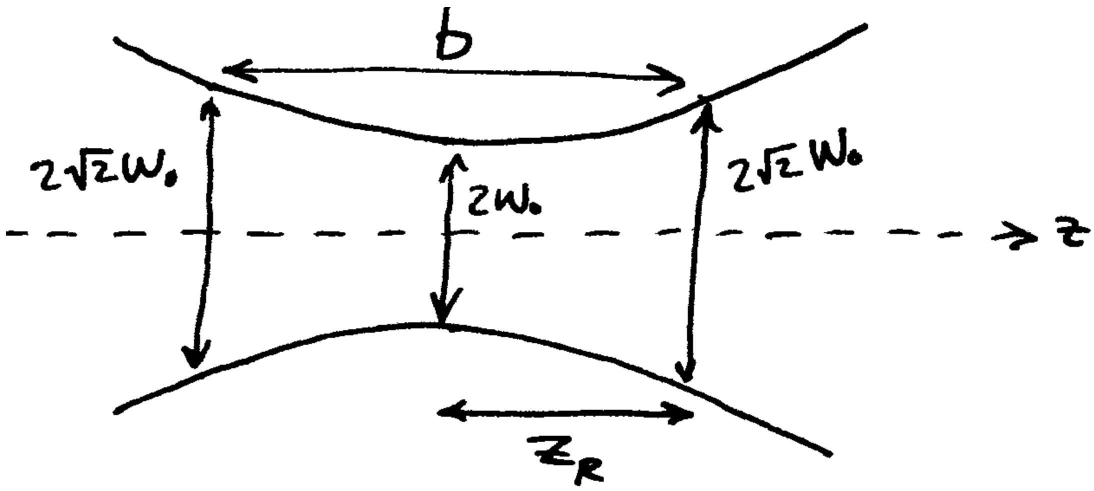
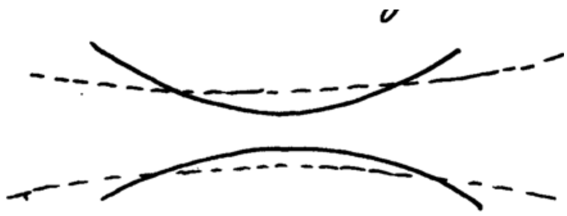
Lecture 33

- Field profile



Note that diffraction is stronger for a smaller spot size:



$\leftarrow$ the smaller the waist , the shorter the Rayleigh range

Beam spreading due to diffraction:

$$w = w_0\sqrt{1+\left(\frac{z}{z_R}\right)^2} \simeq w_0\left(\frac{z}{z_R}\right) \quad \text{when} \quad z \gg z_R \quad \text{(far from waist)}$$

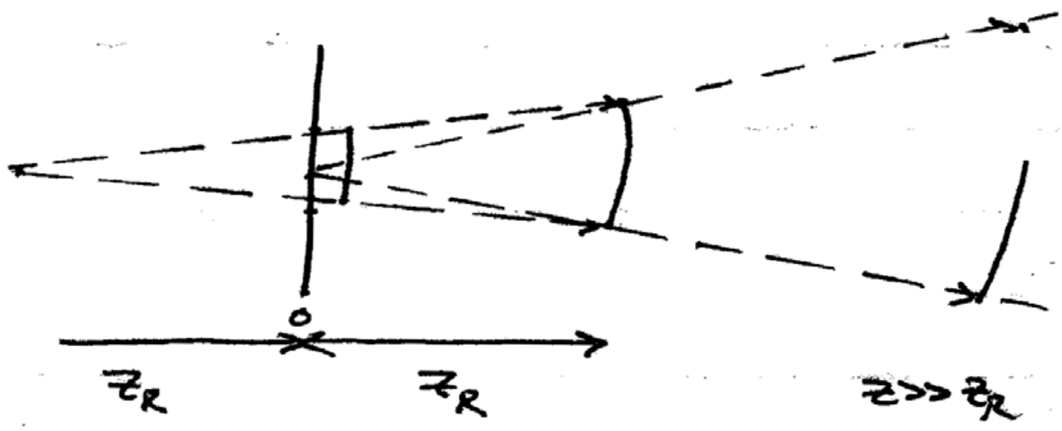
$$\frac{w}{z} = \tan\theta \simeq \theta \simeq \frac{w_0}{z_R} = \frac{w_0}{\pi w_0^2/\lambda} = \frac{\lambda}{\pi w_0}$$

$$\boxed{\theta = \frac{\lambda}{\pi w_0}}$$

- Large w_0 =>small θ and vice versa

Wavefront curvature:

$$R(z) = z + \frac{z_R^2}{z} \simeq \begin{cases} \infty & \text{for } z \ll z_R \text{ (near waist)} \\ 2z_R & \text{at } z = z_R \\ z & \text{for } z \gg z_R \text{ (spherical wave)} \end{cases}$$



Beam power + aperture transmission (Siegman 17.1)

def. $r^2 = x^2 + y^2$

⇨ $\boxed{I(r) = \frac{2\,P}{\pi w^2} e^{-2r^2/w^2}}$

where $P \propto \iint |\varphi|^2 dA =$ total power in beam

equivalent "top-hat" beam with the same peak intensity and the same total power P has diameter $d_{TH} = \sqrt{2} w$

$$A_{TH} = \frac{\pi w^2}{2}$$

- See Siegman fig.17.2.

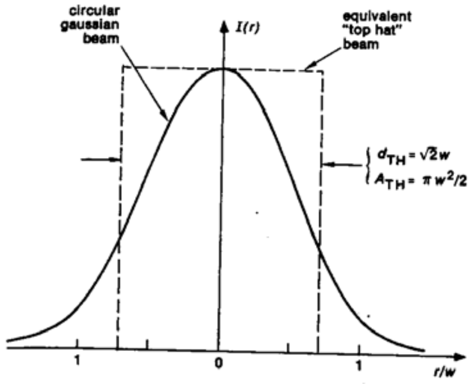


FIGURE 17.2
The equivalent "top hat" radius for a cylindrical gaussian beam.

All the important parameters of this gaussian beam can then be related to the waist spot size w_0 and the ratio z/z_R by the formulas

$$w(z) = w_0\sqrt{1 + \left(\frac{z}{z_R}\right)^2},$$
$$R(z) = z + \frac{z_R^2}{z}, \tag{5}$$
$$\psi(z) = \tan^{-1}\left(\frac{z}{z_R}\right).$$

In other words, the field pattern along the entire gaussian beam is characterized entirely by the single parameter w_0 (or $\tilde{q}_0$, or z_R) at the beam waist, plus the wavelength λ in the medium.

Aperture Transmission

Before exploring the free-space propagation properties of an ideal gaussian beam, we might consider briefly the vignetting effects of the finite apertures that will be present in any real optical system. The intensity of a gaussian beam falls off very rapidly with radius beyond the spot size w. How large must a practical aperture be before its truncation effects on a gaussian beam become negligible?

Suppose we define the total power in an optical beam as $P = \iint |\tilde{u}|^2\, dA$ where dA integrates over the cross-sectional area. The radial intensity variation of a gaussian beam with spot size w is then given by

$$I(r) = \frac{2P}{\pi w^2}\, e^{-2r^2/w^2}. \tag{6}$$

The effective diameter and area of a uniform cylindrical beam (a "top hat beam") with the same peak intensity and total power as a cylindrical gaussian beam will

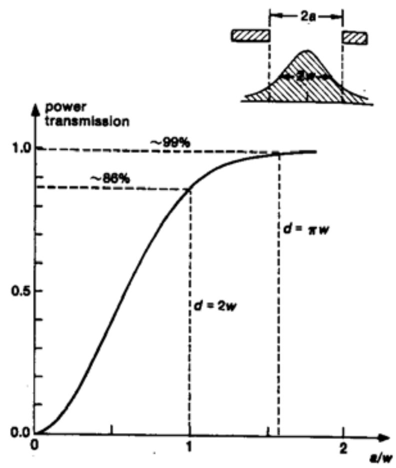


FIGURE 17.3
Power transmission of a cylindrical gaussian beam through a circular aperture.

then be

$$d_{TH} = \sqrt{2}w \quad \text{and} \quad A_{TH} = \frac{\pi w^2}{2} \tag{7}$$

as shown in Figure 17.2.

An aperture significantly larger than this will be needed, however, to pass a real gaussian beam of spot size w without serious clipping of the beam skirts. The fractional power transfer, for example, for a gaussian beam of spot size w passing through a centered circular aperture of diameter $2a$, as in Figure 17.3, will be given by

$$\text{power transmission} = \frac{2}{\pi w^2}\int_0^a 2\pi r e^{-2r^2/w^2}\, dr = 1 - e^{-2a^2/w^2}. \tag{8}$$

This figure plots this transmission versus aperture radius a normalized to spot size w. An aperture with radius $a = w$ transmits $\approx$ 86% of the total power in the gaussian beam. We will refer to this as the $1/e$ or 86% criterion for aperture size.

A more useful rule of thumb to remember, however, is that an aperture with radius $a = (\pi/2)w$, or diameter $d = \pi w$, will pass just over 99% of the gaussian beam power. We will often use this as a practical design criterion for laser beam apertures, and will refer to it as the "$d = \pi w$" or 99% criterion. (A criterion of $d = 3w$ which gives $\approx$ 98.9% transmission would obviously serve equally well.) Figure 17.4 illustrates just where some of these significant diameters for a gaussian beam will fall on the gaussian beam profile.

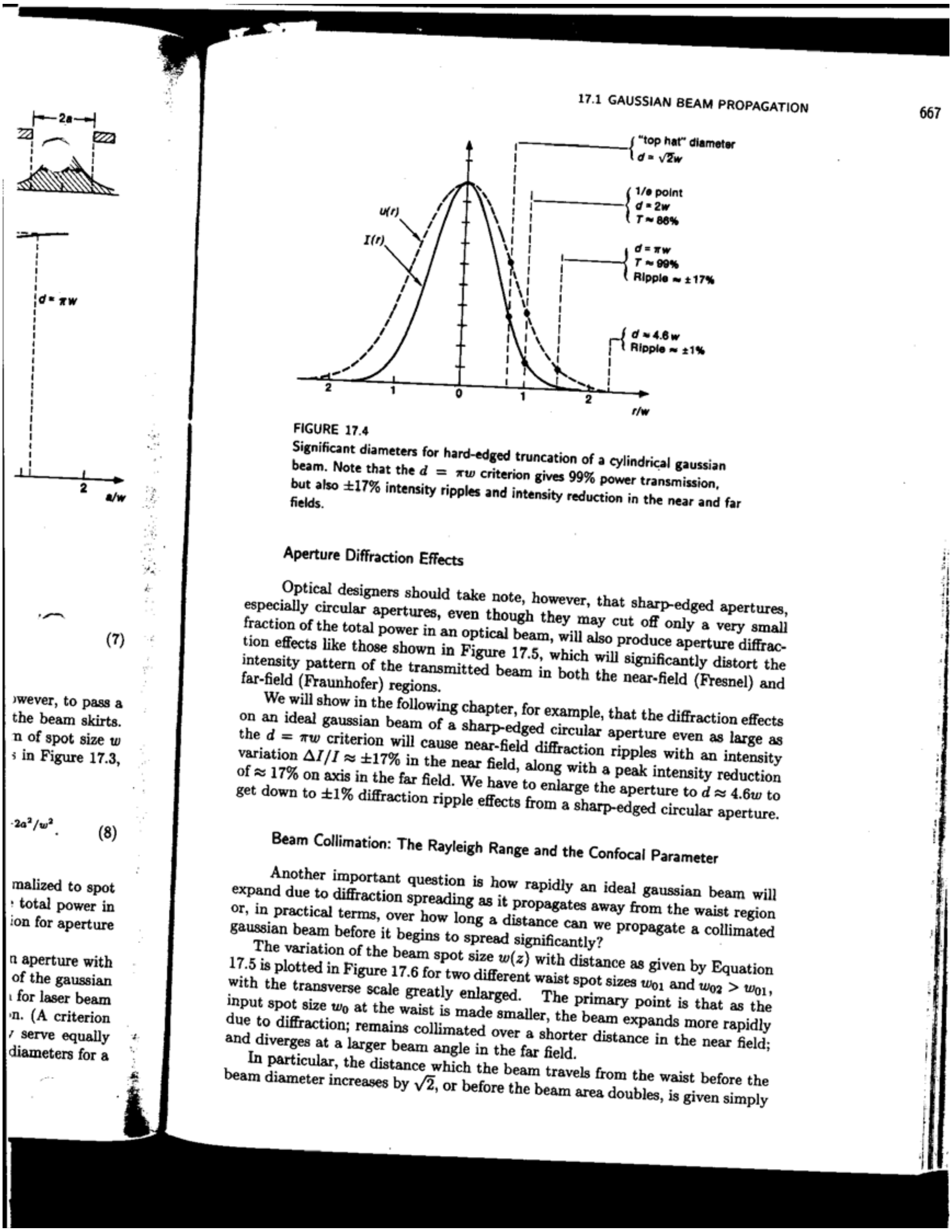


FIGURE 17.4
Significant diameters for hard-edged truncation of a cylindrical gaussian beam. Note that the $d = \pi w$ criterion gives 99% power transmission, but also ±17% intensity ripples and intensity reduction in the near and far fields.

Aperture Diffraction Effects

Optical designers should take note, however, that sharp-edged apertures, especially circular apertures, even though they may cut off only a very small fraction of the total power in an optical beam, will also produce aperture diffraction effects like those shown in Figure 17.5, which will significantly distort the intensity pattern of the transmitted beam in both the near-field (Fresnel) and far-field (Fraunhofer) regions.

We will show in the following chapter, for example, that the diffraction effects on an ideal gaussian beam of a sharp-edged circular aperture even as large as the $d = \pi w$ criterion will cause near-field diffraction ripples with an intensity variation $\Delta I/I \approx \pm 17\%$ in the near field, along with a peak intensity reduction of $\approx 17\%$ on axis in the far field. We have to enlarge the aperture to $d \approx 4.6w$ to get down to ±1% diffraction ripple effects from a sharp-edged circular aperture.

Beam Collimation: The Rayleigh Range and the Confocal Parameter

Another important question is how rapidly an ideal gaussian beam will expand due to diffraction spreading as it propagates away from the waist region or, in practical terms, over how long a distance can we propagate a collimated gaussian beam before it begins to spread significantly?

The variation of the beam spot size $w(z)$ with distance as given by Equation 17.5 is plotted in Figure 17.6 for two different waist spot sizes w_{01} and $w_{02} > w_{01}$, with the transverse scale greatly enlarged. The primary point is that as the input spot size w_0 at the waist is made smaller, the beam expands more rapidly due to diffraction; remains collimated over a shorter distance in the near field; and diverges at a larger beam angle in the far field.

In particular, the distance which the beam travels from the waist before the beam diameter increases by $\sqrt{2}$, or before the beam area doubles, is given simply

For a Gaussian beam passing through a <u>circular aperture</u> the total power getting through is

$$\text{power trans.} = \frac{2}{\pi w^2}\int_0^a 2\pi r e^{-2r^2/w^2}\,dr = 1 - e^{-2a^2/w^2}$$

-see Siegman fig.17.3

dia. d = πw =>get 99% through

However, even though most of the power gets through, significant diffraction effects are seen (as "ripple" on the beam profile – we'll shortly see where that comes from). To get <u>ripple < 1%</u>, <u>need</u>

$\underline{d \geq 4.6w}$ (see Siegman fig.17.4).

<u>Gaussian Beam Propagation</u> + <u>ABCD Matrices</u>

(following Milonni + Eberly 14.6)

We have seen that a Gaussian beam remains Gaussian as it propagates in free space, and that the beam radius and phase front radius of curvature ($\omega(z)$ and $R(z)$ respectively) vary according to the formulas given above.

Now we want to see how q(z)—which contains both w and R – changes on propagation.

Recall $\frac{1}{q(z)} = \frac{1}{R(z)} - \frac{i\lambda}{\pi w^2(z)}$

$q(z) = z + iz_R$ ="complex radius" of Gaussian beam

1. Free space propagation

   If $q(z_1) = q_1$, then at a position $z_2 = z_1 + d$

⇨ $q(z_2) = q_2 = q_1 + d$

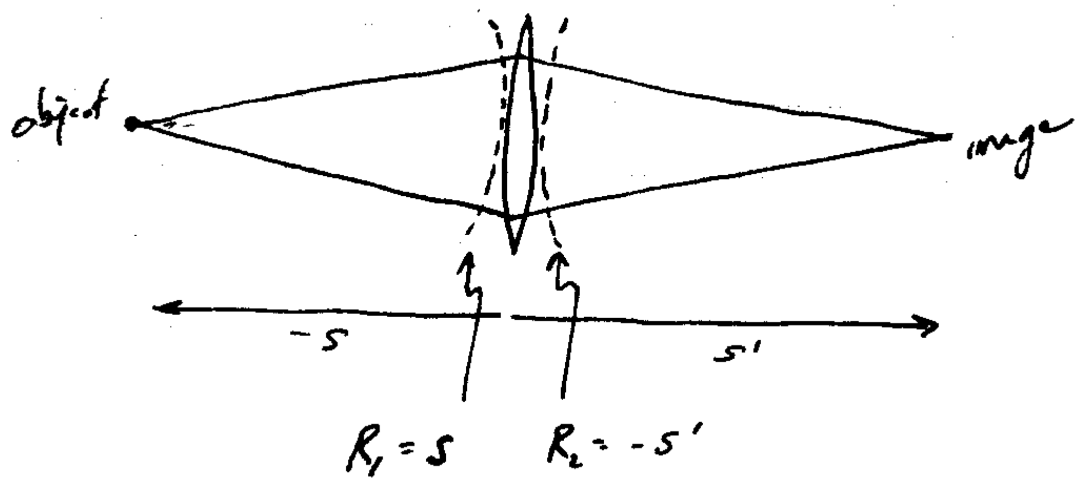
Note that this could be written (with malice aforethought)

$$q_2 = \frac{Aq_1 + B}{Cq_1 + D}$$

Where

$\begin{bmatrix} A & B \\ C & D \end{bmatrix} = \begin{bmatrix} 1 & d \\ 0 & 1 \end{bmatrix}$ = free-space ray matrix

2. Propagation through a thin lens



A Gaussian beam incident from the left has wavefront radius R_1, which is the same as a spherical wave emanating from the object point at a distance S would have.

The lens (being with) will not change the beam radius

⇨ $w_2 = w_1$

The beam will have a wavefront radius R_2 on the right. By the imaging law, it will have the same radius as a spherical wave converging to the image point at a distance S', where

$$\frac{1}{S'} - \frac{1}{S} = \frac{1}{f}$$

Sign convention: $R_1 > 0, R_2 < 0$ as shown

$$-\frac{1}{R_2} + \frac{1}{R_1} = \frac{1}{f}$$

$$\frac{1}{R_2} = \frac{1}{R_1} - \frac{1}{f}$$

Relating the q-parameters,

$$\frac{1}{q_1} = \frac{1}{R_1} - \frac{i\lambda}{\pi w_1^2}, \frac{1}{q_2} = \frac{1}{R_2} - \frac{i\lambda}{\pi w_1^2}$$

$$w_2 = w_1$$

Thus $-\frac{1}{R_2} + \frac{1}{R_1} = -\frac{1}{q_2} + \frac{1}{q_1} = \frac{1}{f}$

$$\frac{1}{q_2} = \frac{1}{q_1} - \frac{1}{f}$$

⇨ $$q_2 = \frac{1}{\frac{1}{q_1} - \frac{1}{f}} = \frac{q_1}{-\frac{q_1}{f} + 1} = \frac{Aq_1 + B}{Cq_1 + D}$$

Where

$$\begin{bmatrix} A & B \\ C & D \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ -\frac{1}{f} & 1 \end{bmatrix}$$

Which is just the ABCD ray matrix for a thin lens.

We have only dealt with two examples, but the fact that the Gaussian beam wavefronts are spherical allows the same argument we gave for transformation of wavefronts on P.214 to be given for Gaussian beams.

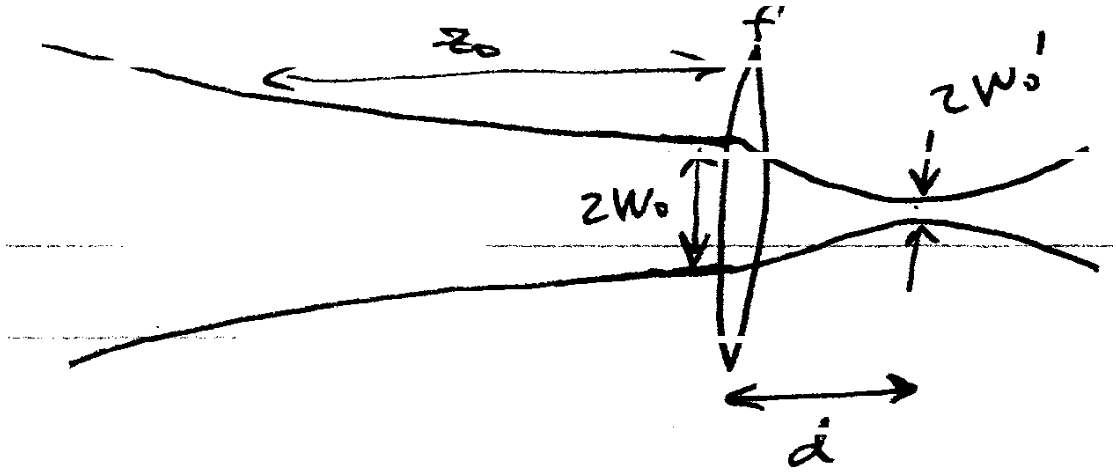
Thus the transformation of a Gaussian beam by an optical system is given by

$$\boxed{q_2 = \frac{Aq_1 + B}{Cq_1 + D}}$$

Where $\begin{bmatrix} A & B \\ C & D \end{bmatrix}$ = ray matrix of geometrical optics

Remember that ABCD matrices were derived for paraxial rays, which form the normals to spherical wavefronts. Since Gaussian beams are paraxial waves, their phase fronts are spherical, and thus transform in the same way.

Example：focusing a Gaussian beam



(usually $z_0 \gg f$)

Incident beam: $\frac{1}{q_1} = \frac{1}{\infty} - \frac{i\lambda}{\pi w_0^2} \Rightarrow q_1 = i\frac{\pi w_0^2}{\lambda}$

Propagation: $\begin{bmatrix} A & B \\ C & D \end{bmatrix} = \begin{bmatrix} 1 & d \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ -\frac{1}{f} & 1 \end{bmatrix} = \begin{bmatrix} 1-\frac{d}{f} & d \\ -\frac{1}{f} & 1 \end{bmatrix}$

$$q_2 = \left[\frac{1}{R_2} - \frac{i\lambda}{\pi w_2^2}\right]^{-1} = \frac{\left(1-\frac{d}{f}\right)\left(\frac{i\pi w_0^2}{\lambda}\right) + d}{\left(-\frac{1}{f}\right)\left(\frac{i\pi w_0^2}{\lambda}\right) + 1}$$

Equate real and imaginary parts:

$$R_2(d)=\frac{\left(\frac{d}{z_0}\right)^2+\left(1-\frac{\mathrm{d}}{f}\right)^2}{\frac{d}{z_0^2}-\frac{1}{f}\left(1-\frac{\mathrm{d}}{f}\right)}$$

$$w_L^2(d)=w_D^2\left(1-\frac{\mathrm{d}}{f}\right)^2+w_D^2\left(\frac{d}{z_0}\right)^2$$

The waist of the focused beam occurs when $R_2(d)=\infty$

$$\frac{d}{z_0^2}-\frac{1}{f}\left(1-\frac{\mathrm{d}}{f}\right)=0$$

$$d\left(\frac{1}{z_0^2}+\frac{1}{f^2}\right)=\frac{1}{f}$$

$$d=\frac{\frac{1}{f}}{\frac{1}{z_0^2}+\frac{1}{f^2}}\Rightarrow\boxed{d=\frac{f}{1+\frac{f^2}{z_0^2}}}$$

Note that if $z_0 \gg f$, then d=f (geometrical optics result).

(i.e. a well-collimated input beam focuses at the "focal length").
At the beam waist, the spot size is

$$w_0'=w\left(d=\frac{f}{1+\frac{f^2}{z_0^2}}\right)=\frac{\lambda f}{\pi w_0}\frac{1}{\sqrt{1+\frac{f^2}{z_0^2}}}$$

Again, note that for $z_0 \gg f$,

$$\boxed{w_0'=\frac{\lambda f}{\pi w_0}}$$

Define beam diameters, $D_0=2w_0, D_0'=2w_0'$

$$D_0'=2w_0'=\frac{4\lambda f}{\pi\left(2w_0\right)}=\frac{4}{\pi}\lambda\left(\frac{f}{D_0}\right)$$

Define <u>f-number</u> $f^{\#}\equiv\frac{f}{D_0}$

(note that this is the f-number determined by the input beam diameter, NOT the lens diameter!)

$$\frac{4}{\pi}=1.27 \Rightarrow \boxed{D_0'=1.27\lambda f^{\#}}$$

Recall diffraction of a plane wave by a circular aperture => $\frac{D}{2}=1.22\lambda f^{\#}$ (we will show this shortly); they are very close to a factor of 2 difference.

Ultimate focusing: the fastest lenses of sufficient quality to give "diffraction –limited" performance have $f^{\#} \le 1 \Rightarrow$

$\boxed{D'_{0\ ultimate} \sim \lambda}$ Smallest achievable focal spot

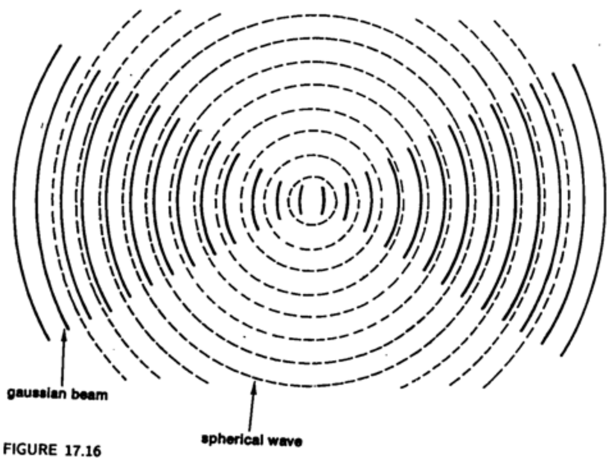


FIGURE 17.16
Alternative picture of the Gouy phase shift through the waist region, as compared to an ideal spherical wave.

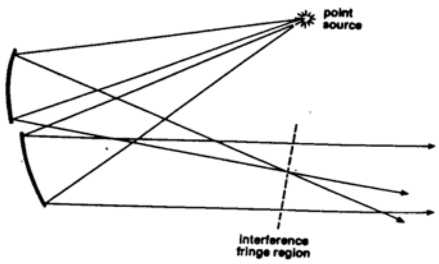


FIGURE 17.17
Experimental apparatus used by Gouy to demonstrate the extra 180° phase shift for an optical beam passing through a focus.

1890, says that a beam with *any* reasonably simple cross section will acquire an extra half-cycle of phase shift in passing through a focal region.

Figure 17.17 shows the simple apparatus employed by ~~Guoy~~ Gouy to demonstrate this effect. In the original experiment the light diverging from a small pinhole was reflected into two overlapping beams reflected from both a planar and a curved mirror. Interference effects between the two beams then produced a set of circular interference fringes between the two beams which could be observed at transverse planes near the first image of the pinhole. Gouy noticed that the centermost fringe in this "bulls-eye pattern" changed sign from dark to light (or vice versa) if he observed the fringes at observation planes just before or just after the focal point. This change of sign implied that the focused beam had somehow picked up an extra π phase shift in passing through the focus.

We will see shortly that higher-order transverse modes, because they have more complicated transverse second derivatives in Equation 17.39, have larger

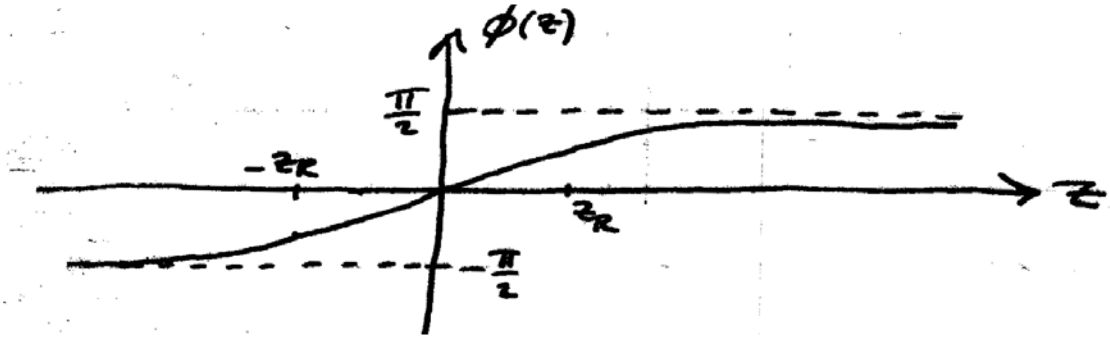
The Gouy Effect

Back to page 336, we found that the electric field had a plane-wave-like phase factor

$$e^{-ikz+i\phi(z)}$$

Where $\tan\phi(z)=\frac{z}{z_R}, z_R=\frac{\pi w_0^2}{\lambda}$

As usual, we put the beam waist at the origin, so that the phase ϕ vs. propagation is



$\rightarrow$ Thus there is a $\underline{\pi}$ phase shift in the wave as it passes through focus, relative to the phase an ideal plane wave would have. This phase shift is known as the Gouy effect.

- See Siegman fig.17.16 : wave fronts shift forward by $\lambda/2$ when going through focus
- Mathematically, we can see how this arises by writing the paraxial wave eqn. as

$$\frac{\partial\varphi}{\partial z}=-\frac{i}{2k}\nabla_T^2\varphi$$

A plane wave would have $\nabla_T^2\varphi=0\Rightarrow$ there would be no excess phase shift.

A beam, however, has a field which is confined in the transverse direction, so $\nabla_T^2\varphi\neq 0 \Rightarrow \varphi$ "accumulates" an additional phase with propagation. It accumulates the most phase where $\nabla_T^2\varphi$ is largest, i.e. within a Rayleigh range or two of the focus.

- See also Feng+Winful, opt lett.26,485(2001)