

0.1 Power spectrum

A spectrometer of course does not measure $\varepsilon(\omega)$, just as a photodetector does not measure $\varepsilon(t)$. They use 'square-law' detectors to measure

$$I(\omega) \propto \tilde{\varepsilon}^*(\omega)\tilde{\varepsilon}(\omega)$$

$$\begin{aligned} I(\omega) &\sim e^{-\frac{a}{2} \frac{(\omega_0 - \omega)^2}{a^2 + b^2}} \text{ (Gaussian)} \\ &= e^{-(4 \ln 2) \frac{(\omega_0 - \omega)^2}{(\Delta\omega_p)^2}} \end{aligned}$$

where $\Delta\omega_p$ is the power spectrum FWHM

$$\Delta\omega_p = 2\sqrt{2 \ln 2} \sqrt{a[1 + (\frac{b}{a})^2]}$$

Note that as the 'chirp parameter' b is increased, the spectral width is also increased. In other words, if you start with an unchirped pulse with fixed duration $\tau = \frac{1}{\sqrt{a}}$, then as chirp is added ($b \neq 0$), the spectral bandwidth of the pulse increases.

def. time-bandwidth product

$$\begin{aligned} \Delta f_p \cdot \tau_p &= \frac{2 \ln 2}{\pi} \sqrt{1 + (\frac{b}{a})^2} \text{ (using } \tau_p = \sqrt{\frac{2 \ln 2}{a}}) \\ &\simeq 0.44 \sqrt{1 + (\frac{b}{a})^2} \text{ (for Gaussian pulses)} \end{aligned}$$

The time-bandwidth product is a **minimum** for $b = 0$ (no chirp). Such a pulse is called **transform-limited**.

From another point of view, suppose you could independently measure $I(t)$ and $I(\omega)$. Then if

$$\mathcal{F}[\sqrt{I(t)}] = \sqrt{I(\omega)}$$

then the pulse is transform-limited. (The proof is left as a homework exercise).

If you measured $I(t)$ and $I(\omega)$, and the pulse had any chirp on it, then the spectral width will be **larger** than that of a pulse of the same length τ_p ; that is why we say it is not transform-limited. In the language of Fourier transforms, the chirped pulse requires more frequency components to make up the pulse than a transform-limited one.

More generally, $\Delta\omega_p$ and τ_p can be seen as Fourier transforms of each other.

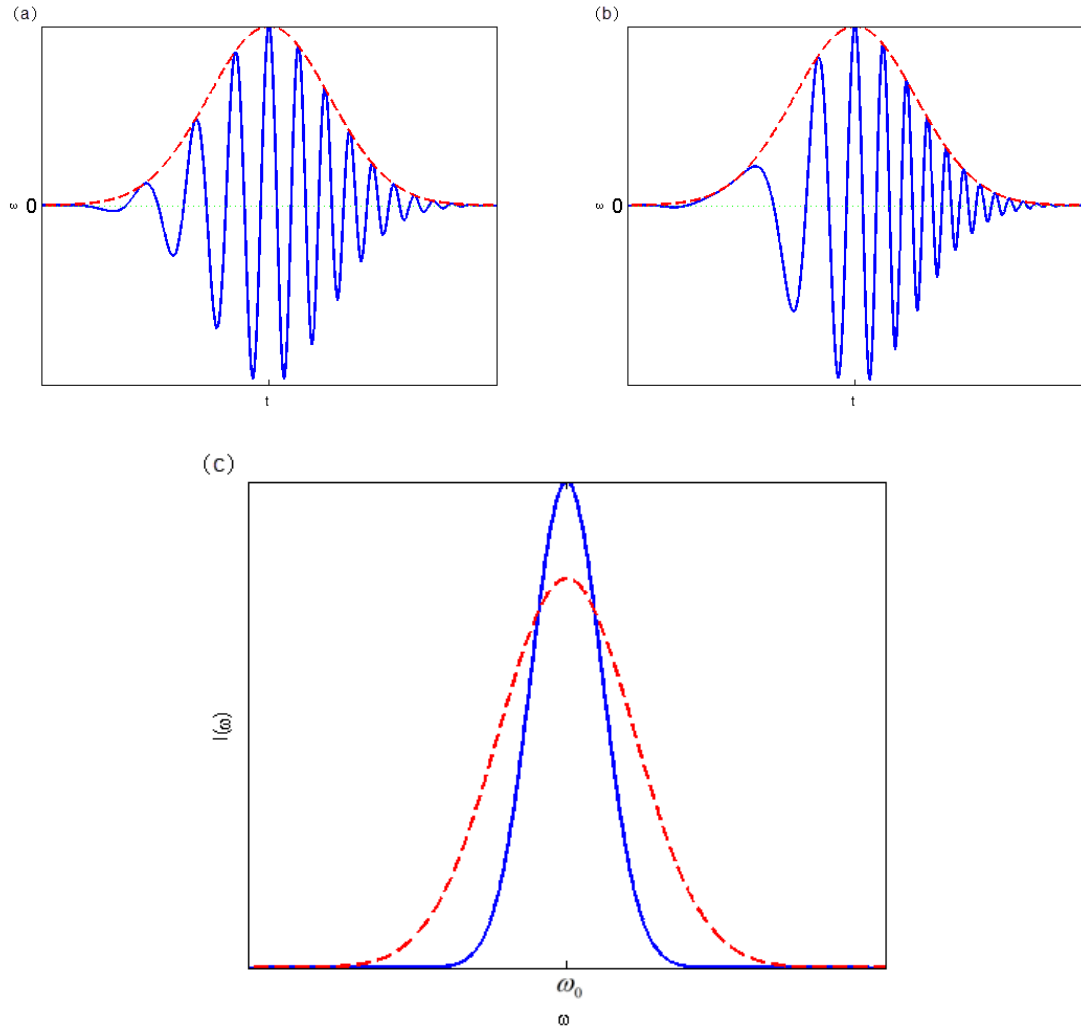


Figure 1: Comparison between cases of different chirp parameter b . In (c), blue solid line corresponds to (a), red dash line corresponds to (b).

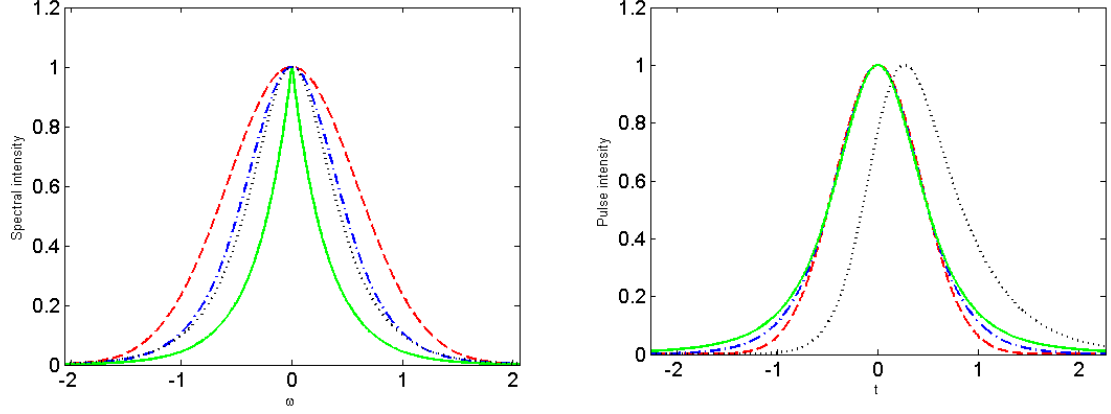


Figure 2: Temporal pulse profiles and the corresponding spectra (normalized).

Time-bandwidth product

$$\Delta\omega_p \cdot \tau_p \geq 2\pi c_B$$

also called the 'uncertainty principle'

where c_B is a numerical constant of order unity, depending on the pulse shape. (note $\Delta f_p \cdot \tau_p \geq c_B$)

Transfer-limited pulses have minimum time-bandwidth product (i.e. equality)

Table 1: Examples of standard pulse profiles.

Field envelope	Intensity profile	$\tau_p(\text{FWHM})$	Spectral profile	$\Delta\omega_p(\text{FWHM})$	c_B
Gauss	$e^{-(t/\tau_p)^2}$	$1.177\tau_G$	$e^{-(\omega\tau_G)^2/2}$	$2.355/\tau_G$	.441
sech	$\text{sech}^2(t/\tau_s)$	$1.763/\tau_s$	$\text{sech}^2(\pi\omega\tau_s/2)$	$1.122/\tau_s$	.315
Lorentz	$[1 + (t/\tau_L)]^{-2}$	$1.287/\tau_L$	$e^{-2 \omega \tau_L}$	$0.693/\tau_L$	.142
asym. sech	$[e^{t/\tau_a} + e^{-3t/\tau_a}]^{-2}$	$1.043/\tau_a$	$\text{sech}(\pi\omega\tau_a/2)$	$1.677/\tau_a$	.278

Note the different shapes and spectra for different functional forms all with the same FWHM τ_p (Figure 2)

Note also that for arbitrary pulse forms, the FWHM may not be defined. Then some other measure of the width must be used (typically the r.m.s.), resulting in a different numerical value of c_B .

0.2 Pulse Propagation

So far, we have considered the temporal and spectral descriptions of optical pulses. Now we want to consider **propagation** of these pulses. At first, we will consider only propagation in **linear, homogeneous, isotropic** media (like glass!). Now, we must consider the spatially-varying electric

field, either in the time or frequency domain:

$$\tilde{\varepsilon}(\omega) \rightarrow \tilde{\varepsilon}(z, \omega), \tilde{\varepsilon}(t) \rightarrow \tilde{\varepsilon}(z, t)$$

linear propagation:

you may recall the general approach from **linear response theory**:

1. given $\tilde{\varepsilon}(z, t)$ at the input to the linear system, decompose into its frequency components via Fourier transformation:

$$\tilde{\varepsilon}(z_i, \omega) = \mathcal{F}[\tilde{\varepsilon}(z_i, t)]$$

2. given the linear response function of the medium, propagate each frequency component through the medium:

$$\tilde{\varepsilon}(z, \omega) = R(\omega) e^{-i\Psi(\omega)} \tilde{\varepsilon}(z_i, \omega)$$

where $R(\omega)$ is (real) amplitude response (describes linear **loss**), $\Psi(\omega)$ is (real) phase response (describes **dispersion**)

3. find the time-domain field at the output by inverse Fourier transformation:

$$\tilde{\varepsilon}(z, t) = \mathcal{F}^{-1}[\tilde{\varepsilon}(z, \omega)]$$

The main question will therefore be, for propagation in some medium or linear optical system, what is the response function?

For propagation in a linear dielectric, we find the response by considering the Maxwell wave eqn.

$$\left(\frac{\partial^2}{\partial z^2} - \mu_0 \epsilon_0 \frac{\partial^2}{\partial t^2} \right) \varepsilon(z, t) = \mu_0 \frac{\partial^2 p}{\partial t^2}$$

(For now we will consider only plane waves, so we can ignore the transverse Laplacian $\nabla_t^2 = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}$, which is responsible for diffraction of finite-size beams. More on that later!)

As usual, for a linear medium one wants to start with a polarization $p = \epsilon_0 \chi \varepsilon$. But, we need to be careful. In the time domain, the field induces a polarization, which acts as a source term, modifying the field, and so on. Thus the field (and the polarization) depend on their past history, which can be highly nontrivial. This makes a direct time-domain solution for an arbitrary propagation problem

difficult. In the time domain

$$p(z, t) = \epsilon_0 \int_{-\infty}^t \chi(t') \varepsilon(z, t - t') dt'$$

where $\chi(t')$ is the (casual) **linear susceptibility**, i.e. the **impulse response** of the medium.

Much easier: in the frequency domain we have simply

$$\tilde{p}(z, \omega) = \epsilon_0 \tilde{\chi}(\omega) \tilde{\varepsilon}(z, \omega)$$

(Note that $\tilde{p}(\omega)$ and $p(t)$ are just Fourier transforms of each other.) Thus all we need to do is to solve the wave eqn. for harmonic waves, and the temporal response is obtained by summing up all the harmonics (i.e. Fourier transforming).

We need to only consider $\tilde{\varepsilon}(z, \omega) = \tilde{E}(z, \omega) e^{i\omega t}$, $\tilde{E}(z, \omega)$ is envelope function in frequency domain.

The source term is

$$\begin{aligned} \frac{\partial^2}{\partial t^2} \tilde{p} &= \epsilon_0 \tilde{\chi}(\omega) \frac{\partial^2}{\partial t^2} \tilde{\varepsilon} \\ &= \epsilon_0 \tilde{\chi}(\omega) \tilde{E}(z, \omega) (-\omega^2 e^{i\omega t}) = -\omega^2 \epsilon_0 \tilde{\chi} \tilde{\varepsilon} \end{aligned}$$

The wave eqn. becomes

$$\left(\frac{\partial^2}{\partial z^2} - \mu_0 \epsilon_0 \frac{\partial^2}{\partial t^2} \right) \tilde{\varepsilon} = -\omega^2 \mu_0 \epsilon_0 \tilde{\chi} \tilde{\varepsilon}$$

$$\left(\frac{\partial^2}{\partial z^2} + \mu_0 \epsilon_0 \omega^2 \right) \tilde{E} = -\omega^2 \mu_0 \epsilon_0 \tilde{\chi} \tilde{E}$$

$\Rightarrow$ Helmholtz eqn.

$$\left[\frac{\partial^2}{\partial z^2} + \frac{\omega^2}{c_0^2} (1 + \tilde{\chi}) \right] \tilde{E}(z, \omega) = 0, \quad c_0^2 = \frac{1}{\mu_0 \epsilon_0}$$

As usual, we define the complex dielectric constant

$$\tilde{\epsilon} = 1 + \tilde{\chi}(\omega)$$

and index of refraction

$$\tilde{n} = \sqrt{\tilde{\epsilon}}$$

0.3 Review: Lorentz model

For those not completely comfortable with this model, see Siegman Lasers chapter 2. Basically, it gives the simplest possible model for the susceptibility, namely that the material response can be modeled as a simple harmonic oscillator. In general, determining $\tilde{\chi}$ from first principles requires a quantum-mechanical calculation of the atomic response to an applied field. The classical model works quite well, however, if we accept that certain parameters (resonant frequency, damping, and oscillator strength) must be taken from experiment. (The precise relation between the two models is treated in depth in 539.)

Skipping many important details, the essential idea is to consider the dielectric to be composed of electrons on springs:

$$F = ma$$

$$-\frac{e}{m}\varepsilon - \omega_a^2 x - \gamma \frac{dx}{dt} = \frac{d^2 x}{dt^2}$$

$-\frac{e}{m}\varepsilon$: external field; $-\omega_a^2 x$: spring; $-\gamma \frac{dx}{dt}$: damping.

The induced polarization is

$$p = -Ne\chi, \quad N = \text{number of oscillators per volume}$$

$\Rightarrow$

$$\frac{Ne^2}{m}\tilde{\varepsilon} = \omega_a^2 \tilde{p} + \gamma \frac{d\tilde{p}}{dt} + \frac{d^2 \tilde{p}}{dt^2}$$

If the driving field is harmonic, so will the polarization

$$\tilde{\varepsilon} = \tilde{E}e^{i\omega t}, \quad \tilde{p} = \tilde{P}e^{i\omega t}$$

$\Rightarrow$ the amplitudes obey

$$\frac{Ne^2}{m}\tilde{E} = \omega_a^2 \tilde{P} + i\omega\gamma \tilde{P} - \omega^2 \tilde{P}$$

$\Rightarrow$

$$\tilde{P} = \frac{\frac{Nfe^2}{m}}{\omega_a^2 - \omega^2 + i\omega\gamma} \tilde{E} = \epsilon_0 \tilde{\chi} \tilde{E}$$

where $\tilde{\chi} = \frac{Nfe^2}{m\epsilon_0} \frac{1}{\omega_a^2 - \omega^2 + i\omega\gamma}$, and we have introduced a 'fudge?? factor' f , called the **oscillator strength**, which is required to model real dielectric media (can be calculated in quantum theory).

As you recall, the complex susceptibility is responsible for both **loss** (or gain) via the **imaginary** part, and **dispersion** (frequency-dependent index of refraction) via the real part. In laser gain or saturable absorber media, the resonant susceptibility must be considered, i.e. $\omega \simeq \omega_a$. We will return to the discussion of this case in treating mode-locked lasers. For the moment, we are considering only propagation in dielectrics **far from any resonances** (e.g. visible light in glass).

For $\omega \ll \omega_a$ or $\omega \gg \omega_a$, γ can be neglected, and $\tilde{\chi}$ is real (no loss), $\tilde{\chi} = \frac{Nfe^2}{m\epsilon_0} \frac{1}{\omega_a^2 - \omega^2}$

$$n^2 = 1 + \chi = 1 + \frac{Nfe^2}{m\epsilon_0} \frac{1}{\omega_a^2 - \omega^2}$$

Sometimes this is expressed in terms of wavelength:

$$\frac{1}{\omega_a^2 - \omega^2} = \frac{1}{(2\pi\epsilon_0)^2} \frac{\lambda^2 \lambda_a^2}{\lambda^2 - \lambda_a^2}$$

Plugging this in the expression for n^2 above yields the **Sellmeier equation**.

Now that we have $n^2(\omega)$, we can go back to the Helmholtz equation:

$$[\frac{\partial^2}{\partial z^2} + \frac{n^2 \omega^2}{c_0^2}] \tilde{E}(z, \omega) = 0$$

or

$$[\frac{\partial^2}{\partial z^2} + \beta^2] \tilde{E} = 0$$

where $\beta = \frac{n\omega}{c_0}$ =propagation constant.

The solutions are obvious:

$$\tilde{E}(z, \omega) = E_0 e^{-i\beta z}$$

for a wave propagating in the +z direction.

Therefore the **material response function** we were looking for is just

$$e^{-i\beta(\omega)z}, \beta(\omega) = \frac{n(\omega)\omega}{c_0}$$

Note that β is a fairly complicated function of ω . While it is not difficult to calculate numerically pulse propagation using the complete expression for $\beta(\omega)$, we can only proceed analytically and gain any physical insight if we make some simplifications.

Note: when considering pulse propagation in **waveguides**, then the relevant propagation constant β is not the material constant $\frac{n(\omega)\omega}{c_0}$, but is the β determined from the waveguide eigenvalue problem.

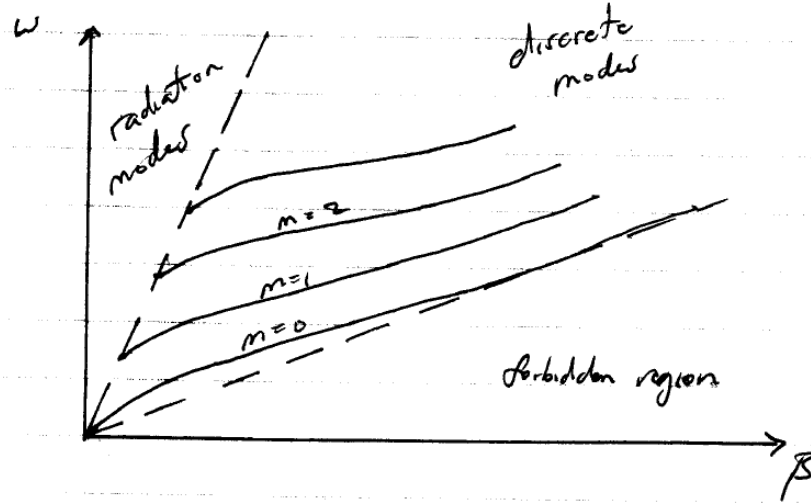


Figure 3: Typical waveguide dispersion diagram. (see 537 notes p. A19.1)

We still have transfer function?? $e^{-i\beta(\omega)z}$, where $\beta(\omega)$ is waveguide propagation constant.

0.4 example: Gaussian pulse propagation in a dispersive medium

(This follows Segman p.336)

input pulse:

$$\tilde{\varepsilon}_0(t) = e^{-\Gamma_0 t^2} e^{i\omega_0 t}$$

$$\tilde{E}_0(\omega) = e^{-\frac{(\omega_0 - \omega)^2}{4\Gamma_0}}$$

$\Gamma_0 = a_0 - ib_0$. After propagating a distance z , the frequency-domain field will be

$$\tilde{E}(z, \omega) = \tilde{E}_0(\omega) e^{-i\beta(\omega)z}$$

Approximation:

Suppose that in the vicinity of the carrier frequency ω_0 , the propagation factor $\beta(\omega)$ varies slowly (Figure 4), and the Taylor expansion to the second order gives

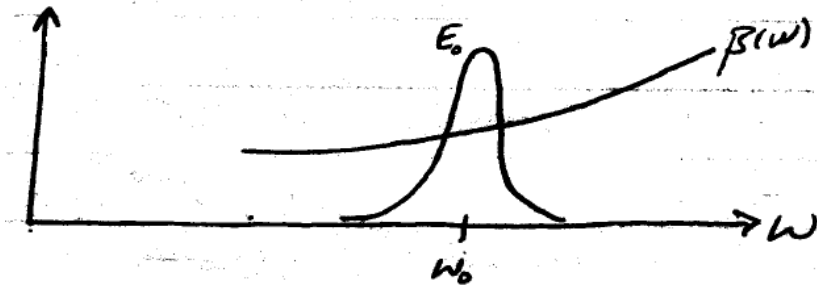


Figure 4: $\beta(\omega)$ varies slowly around ω_0 .

$$\beta(\omega) \simeq \beta(\omega_0) + \frac{d\beta}{d\omega}|_{\omega_0}(\omega - \omega_0) + \frac{1}{2} \frac{d^2\beta}{d\omega^2}|_{\omega_0}(\omega - \omega_0)^2$$

$$= \beta(\omega_0) + \beta'(\omega - \omega_0) + \frac{1}{2}\beta''(\omega - \omega_0)^2$$