Lecture 36

$$E(p_0) = +i\frac{E_0}{\lambda}\cdot\frac{e^{-ikz}}{z}\iint_\Sigma f(x,y)e^{-ik\left[(x-\xi)^2+(y-n)^2\right]/2z}dxdy$$

This is the Fresnel diffraction integral. It says that the wave at P_0 looks like a spherical wave emitted from the center of the aperture, modified by the integral which now has a quadratic (or parabolic) phase factor in it. Remember our paraxial approximation to a spherical wave gave us a parabolic wave before.

We shall return to a discussion of the Fresnel diffraction problem shortly. A large number of diffraction problems can be solved with further simplifying approximation to the integral.

Fraunhofer Diffraction

Consider the phase form

$$\frac{(x-\xi)^2+(y-\eta)^2}{2z} = \frac{(x^2+y^2)}{2z} - \left(\frac{x\xi-y\eta}{z}\right) + \frac{(\xi^2+\eta^2)}{2z}$$

The Fraunhofer, or far-field approximation, consists of neglecting the first term:

$$z \gg \frac{k\ (x^2+y^2)}{2}$$

Let $W = \max(x, y)$ in aperture (max. radius)

$$z \gg \frac{kW^2}{2} = \frac{\pi W^2}{\lambda}$$

$$\boxed{z \gg \frac{\pi W^2}{\lambda}}$$

Note the similarity to the Gaussian beam

Rayligh image $Z_R = \dfrac{\pi W_0^2}{\lambda}$!

⇨ Observation is in far field if it is
"many Rayleigh ranges" away from aperture

Some numbers:

$\lambda = 600nm, (x^2+y^2)_{max} = 1$ inch (that's big !)

$\rightarrow z \gg 1600m$ required! $\boxed{W = 100um \rightarrow z \gg 5cm}$

Thus in practice, for large apertures the Fraunhofer condition is not easily satisfied. Nevertheless, Fraunhefer diffraction is extremely important in many (if not most) situations, since lenses may be used to effectively put the source and observer at infinity. To see this, rewrite the diffraction integral as

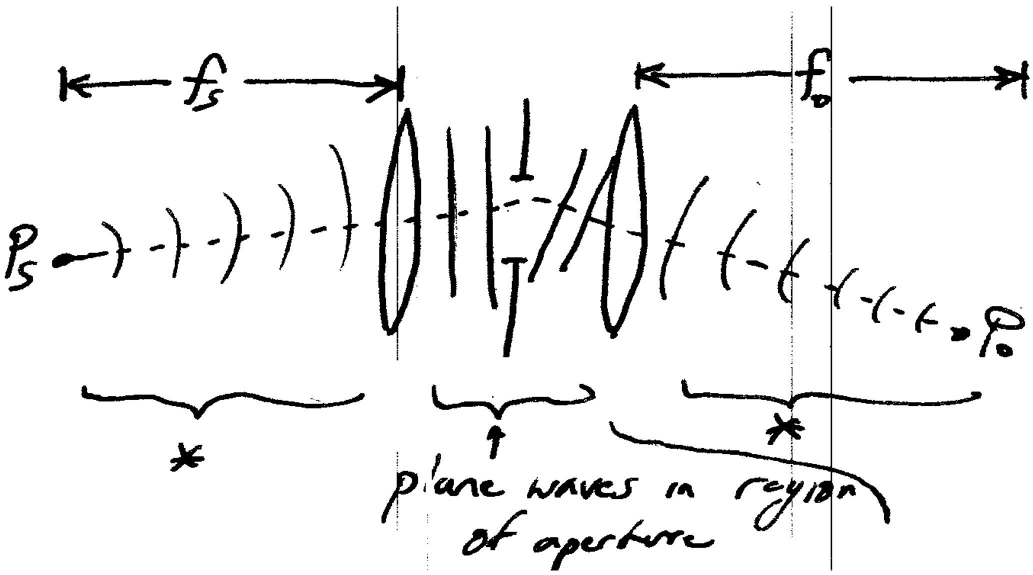
$$\varphi(p_0) = \underbrace{\varphi(\xi,\ \eta) = +i\frac{E_0}{\lambda}\cdot\frac{e^{-ikz}}{z}e^{-ik(\xi^2+\eta^2)/2z}\iint_\Sigma f(x,y)e^{ik(x\xi+y\eta)/2z}dxdy}_{\text{Fraunhefer diffraction integral}}$$

The crucial thing to note is that the phase term in the integral is linear in the aperture variables (x,y).

This is the defining feature of Fraunhofer diffraction.

It means that the wavefronts in the aperture are essentially plane waves.

And, as we know , we can get plane waves from point sources with lenses :



- Spherical waves diverging /converging from/forwards source/observation points.

It is useful to write the Faunhofer integral in terms of some real variables,

Define $w_x = -\frac{k\xi}{z}, w_y = -\frac{k\eta}{z}$

(The reason for the minus sign will be clear momentarily.)

Note that the units of $w_{x,y}$ are the same as for k, i.e. inverse length .

For this reason, w_x and w_y are often called spatial frequencies .In terms of these,

$$\mathrm{E(P_0)} = +i\frac{E_0}{\lambda}\cdot\frac{e^{-ikz}}{z}e^{-ik(\xi^2+\eta^2)/2z}\iint_\Sigma f(x,y)e^{-i(w_x x+w_y y)}dxdy$$

In fact, since the aperture function f(x,y) includes the effect of the aperture, the limits on the integral can be extended to $\pm\infty$, and we have

$$\iint_{-\infty}^{\infty} f(x,y)e^{-i(w_x x+w_y y)}dxdy$$

This is just a two-dimensional Fourier transform!

In fact, in most experiments we simply want the intensity distribution in the observation plane, so we can ignore the phase factors out front and state that

The Fraunhofer diffraction pattern (intensity) is the (square of the) 2-D Fourier transform of the aperture function.

In fact, it is this realization which under pins the entire field of "Fourier optics"
(Note that, following Guenther, we put the minus sign in the definition of the spatial frequencies in order to get the same form of Fourier transform equation as we had with time-frequency transforms.)

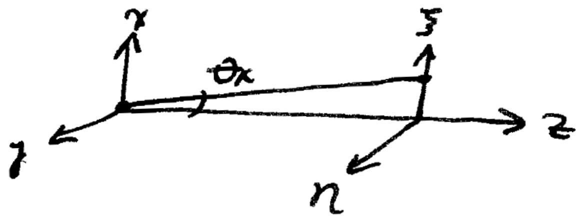
Note on interpretation of spatial frequencies:
e.g.

$$w_x = -\frac{k\xi}{z} = -\frac{2\pi}{\lambda}\frac{\xi}{z}$$

Def

$$\tan\theta_x = \frac{\xi}{z}$$

$$\sin\theta_x \simeq \tan\theta_x \simeq \theta_x = \frac{\xi}{z}$$



And we have $w_x = -\frac{2\pi}{\lambda}\theta_x$

Similarly $w_y = -\frac{2\pi}{\lambda}\theta_y$

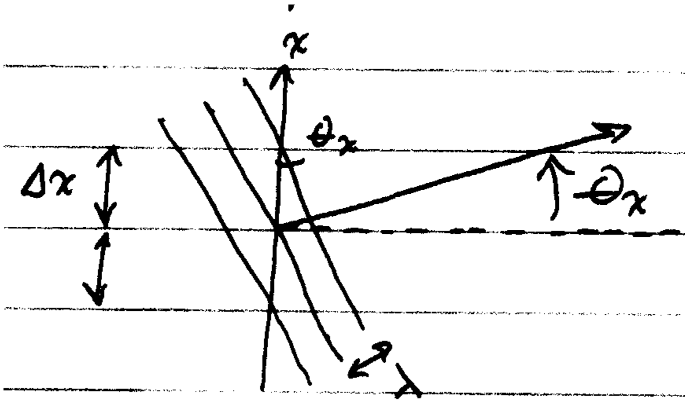
Thus the angular spatial frequencies essentially measure the angle from the aperture to the observation point (in units of the wavevector).

$\theta_{x,y} \simeq 0 \leftrightarrow$ "low spatial frequencies" (near center of diffraction pattern)

$\theta_{x,y}$ large $\leftrightarrow$" high spatial frequencies"(near edge of diffraction pattern)

- The fraunhofer diffraction pattern is thus the Fourier decomposition of the spatial frequencies required to "contract" the aperture function f(x,y).

To see this another way, consider a plane wave in the aperture propagating at an angle θ_x (i.e. consider just one "spatial frequency component" of the total field in the aperture) , and look at the spacing of the wavefronts in the aperture:



Spacing = Δx

Geometry: $\sin\theta_x \simeq \theta_x = \dfrac{\lambda}{\Delta x}$

$$|\Delta x| = \frac{\lambda}{|\theta_x|} = \frac{\lambda}{\frac{2\pi}{\lambda} w_x} = \frac{2\pi}{w_x}$$

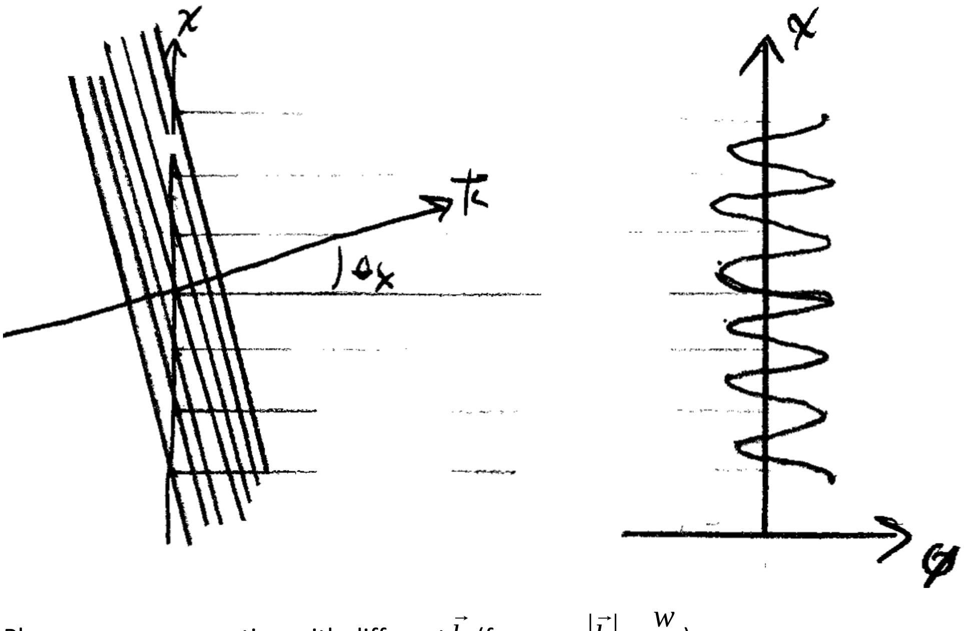
⇨ The "spatial frequency" in the aperture in "lines per millimeter" (common units) is

$$\frac{1}{\Delta x} = \frac{w_x}{2\pi}$$

i.e. w_x is just 2 π x the inverse period of a "grating" in the aperture plane

Any arbitrary aperture function f(x,y) can be decomposed into its spatial frequency Fourier components.

Note that if we plot the field amplitude in the aperture for a plane wave with spatial frequency w_x (i.e. angle= θ_x), we have a sinusoidal function.



Plane waves propagating with different $\vec{k}$ (for same $|\vec{k}| = \dfrac{w}{c}$)

i.e. at different angles θ_x, thus provide a basis set for Fourier expression of the aperture function f(x,y).

Recall our discussion (p.86) of the <u>angular spectrum representation</u> of the field in a given plane z

$$E(x,y,z)=\iint_{-\infty}^{\infty} A\left(k_x,k_y\right)e^{-i\left(k_x x+k_y y\right)}dk_x dk_y$$

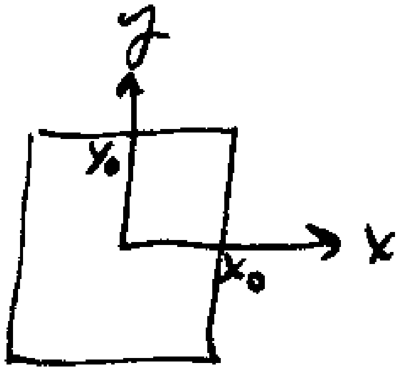
It’s the same physical idea!

(Exercise for reader: relate the angular spectrum representation to the Fraunhofer diffraction integral.)

<u>Example: rectangular aperture</u> with an incident <u>plane wave</u>

Aperture function

$$f\left(x,y\right)=\begin{cases}1, |x|\le x_0, |y|\le |y_0| \\ 0, otherwise\end{cases}$$ “rect function”



$$\varphi\left(\xi,\eta\right)=+i\frac{E_0}{\lambda}\frac{e^{-ikz}}{z}e^{-ik\left(\xi^2+\eta^2\right)/2z}\int_{-x_0}^{x_0} e^{-iw_x x}dx\int_{-y_0}^{y_0} e^{-iw_y y}dy$$

$$\int_{-x_0}^{x_0} e^{-iw_x x}dx=\frac{e^{-iw_x x}}{-iw_x}\Big|_{-x_0}^{x_0}=\frac{e^{-iw_x x}-e^{iw_x x}}{-iw_x}=2x_0\left(\frac{\sin w_x x_0}{w_x x_0}\right)=2x_0\sin w_x x_0$$

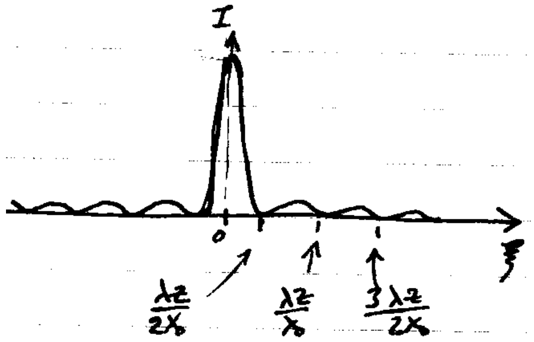
The intensity in the observation plane is therefore of the form

$$I=I_0\left(\sin c^2 w_x x_0\right)\left(\sin c^2 w_y y_0\right)=I_0\left[\sin c^2\left(\frac{kx_0}{z}\xi\right)\right]\left[\sin c^2\left(\frac{ky_0}{z}\eta\right)\right]$$

Recall sinc(0)=1 => $I=I_0$ = max. indensity at center zeroes of $\sin c\alpha$ are at

$\alpha=\pi,2\pi,3\pi,\ldots$

$$\frac{kx_0}{z}\xi=m\pi\ \left(m\neq 0\right)\ \Rightarrow \xi=\frac{\lambda z}{x_0}\frac{m}{2}$$



$\Rightarrow$ width of sinc function = distance between zeros= $\frac{\lambda z}{x_0}$

Large x_0 (wide aperture) $\Rightarrow$ small width of diffraction pattern

+ vice versa (just Fouries transform relation!)

Interpretation in terms of spatial frequencies:

Narrow slit => large distribution of spatial frequencies required to generate the aerure function

=> broad diffraction pattern

(infinity narrow => aperture = δ - function => infinity wide diffraction pattern)

wide slit => primarily low spatial frequencies required to generate aperture function

=> narrow diffraction pattern

(infinity wide => diffraction pattern ~ δ - function !)

Note: slit = rect. Aperture with $x_0 \gg y_0$ (or vice versa)

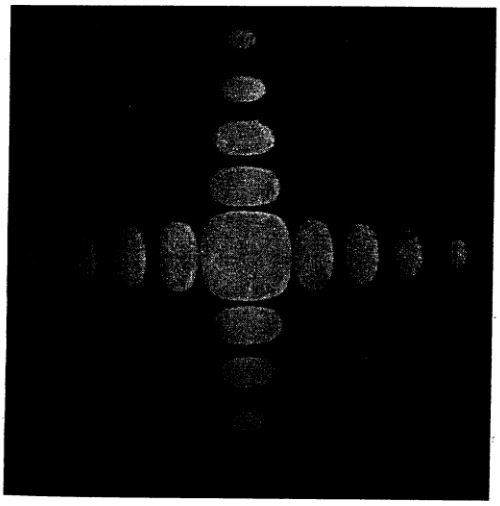


Fig. 8.10 Fraunhofer diffraction pattern of a rectangular aperture 8 mm × 7 mm, magnification 50×, mercury yellow light $\lambda = 5790$ Å. To show the existence of the weak secondary maxima the central portion was overexposed. (Photograph courtesy of H. Lipson, C. A. Taylor, and B. J. Thompson.)

Example: circular aperture illuminated by plane wave

Symmetry => convert to cylindrical coordinates

$(x, y) \to (s, \phi)$ in aperture plane

$(\xi, \eta) \to (\rho, \theta)$ in observation plane

⇨ (see Guenther for the algebra) – dis. Lipson Appendix I

$$\varphi(P_0) = +i\frac{E_0}{\lambda}\frac{e^{-ikz}}{z}\int_0^{a/2}\int_0^{2\pi} f(s,\phi)e^{-ik\frac{s\beta}{z}\cos(\theta-\phi)}sdsd\phi$$

Aperture function

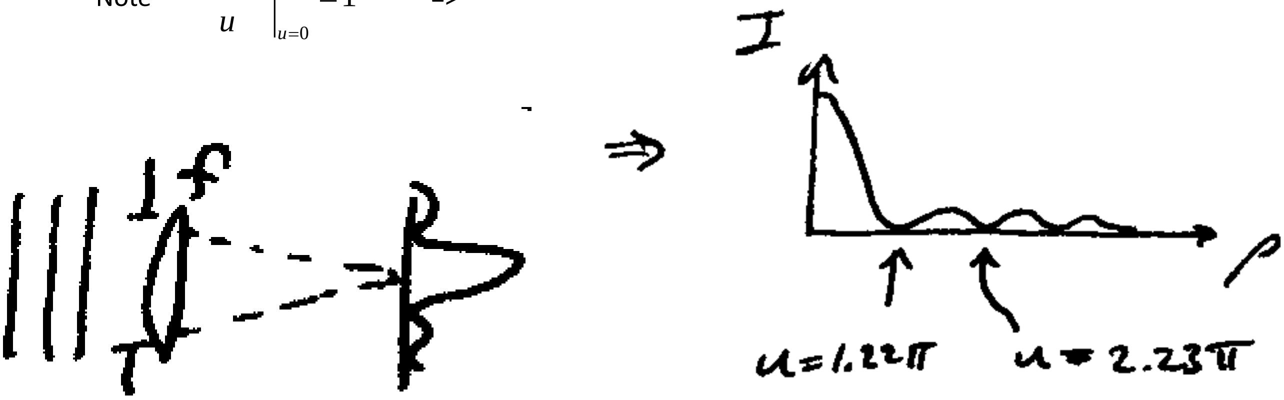
$$f(s,\phi) = \begin{Bmatrix} 1, s \le a/2 \\ 0, s > a/2 \end{Bmatrix}$$

⇨ $$\varphi(\rho,\phi) = +i\frac{E_0}{\lambda}\frac{e^{-ikz}}{z}\left[\frac{\pi a}{k\rho}J_1\left(\frac{ka\rho}{2z}\right)\right]$$

Where $J_1(u) = \frac{u}{2}\left[1 - \frac{1}{1!2!}\left(\frac{u}{2}\right)^2 + \frac{1}{2!3!}\left(\frac{u}{2}\right)^4 - \frac{1}{3!4!}\left(\frac{u}{2}\right)^6 + \ldots\right]$ = Bessel function of first order

$$u = \frac{ka\rho}{2z} \Rightarrow \boxed{I = I_0\left[\frac{2J_1(u)}{u}\right]^2}$$ <u>Airy formula</u>

Note $\left.\frac{2J_1(u)}{u}\right|_{u=0} = 1$ =>



1st minimum is at

$$u_1 = \frac{ka\rho_1}{2f}$$ where z=f=focal length of lens

$$\frac{\pi a\rho_1}{\lambda f} = 1.22\pi$$

$$\Rightarrow \rho_1 = 1.22\left(\frac{f}{a}\right)$$

Recall in the past we have defined the f-number of a lens to be

$$f/\# \equiv \frac{f}{a}$$

Thus we get the famous formula

$$\boxed{\rho_1 = 1.22\lambda f /}$$ ← <u>radius</u> of first minimum

Compare this result to our Gaussian before (exercise for readers)