Lecture 18

We have gone through quite a lot of work to obtain $\tilde{\chi}(t)$ from $\chi(\omega)$, but we have introduced several new concepts which will be useful down the road.

1. complex ω => straightforward description of damping
2. poles of $\chi(\omega)$ are the " normal mode " frequencies
3. causality => impulse response $\propto \theta(t)$
4. note that $\tilde{\chi}(t)$ is real; this turns out to be more generally true (more detailed discussion of this point may be found by the interested reader in Mandel + Wolf's optical (Coherence and Quantum Optics section 3.1)

We have $\vec{P}(t) = \varepsilon_0 \int\limits_{-\infty}^{\infty} \chi(t-t')\vec{E}(t')dt'$

Where $\chi(t-t')$ is the (real) impulse response function.

The frequency domain susceptibility can be written

$$\chi(\omega) = \int\limits_{-\infty}^{\infty} \chi(t) e^{-i\omega t} dt$$

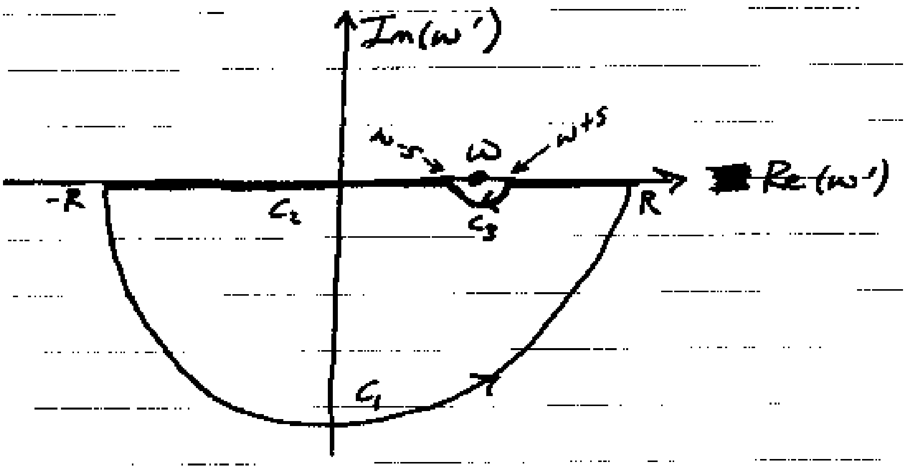
Since $\chi(t)$ is real, $\chi(\omega)$ must be analytic in the lower half in the complex plane. We saw that in the example on P.119, but it is clearly true in general, provided that $\chi(t \to \infty) \to 0$ so that $\chi(\omega)$ will be analytic on the real(ω)axis. I leave it as an exercise to the reader to convince him/herself of this (hint: follow an argument similar to the one we made in evaluating $\int\limits_{C_2}$ on page 120).

Given the simple consequence that $\chi(\omega)$ is an analytic function in the lower half plane, Cauchy's integral formula then says we can write

$$\chi(\omega) = \frac{1}{2\pi i} \oint\limits_C \frac{\chi(\omega')}{\omega' - \omega} d\omega'$$

Where ω is any point in the lower half plane or in the real axis. Note that physically, we are interested in $\chi(\omega)$ for real values of ω. That means the function $\dfrac{\chi(\omega')}{\omega' - \omega}$ has a simple pole on the real axis at the point ω.

We therefore choose the following curve C' of integration:



As we did with our example, we can break the integral up into three parts:

$$\oint_{C'} = \oint_{C_1} + \oint_{C_2} + \oint_{C_3} = 0 \quad \text{Since } \frac{\chi(\omega')}{\omega' - \omega} \text{ analytic over C'}$$

Also as before, we take

$$\int_{-\infty}^{\infty} = \lim_{R\to\infty} \int_{-R}^{R}$$

First consider the integral over C_3

-- integrating around a circle center on $\omega' = \omega$ would give

$$\oint \frac{\chi(\omega')}{\omega' - \omega} d\omega' = 2\pi i \chi(\omega)$$

-- C_3 is only half the circle, and the sense is opposite, so

$$\oint_{C_3} \frac{\chi(\omega')}{\omega' - \omega} d\omega' = -\pi i \chi(\omega)$$

As with our example, $\int_{C_1} \to 0$ as $R \to \infty$ (proof is left to reader).

The remaining integral is along the real axis, and is the principal part, so we have

$$P\int_{-\infty}^{\infty} \frac{\chi(\omega')}{\omega' - \omega} d\omega' + \pi i \chi(\omega) = 0$$

Where P indicate the principal part ($\underline{\underline{P\int_{-\infty}^{\infty} = \lim_{S\to 0}\left\{ \int_{-\infty}^{\omega - S} + \int_{\omega+S}^{\infty} \right\}}}$).We thus have

$$\int_{C_2} = -P\int_{-\infty}^{\infty}$$

$$\chi(\omega) = \frac{-1}{\pi i} P\int_{-\infty}^{\infty} \frac{\chi(\omega')}{\omega' - \omega} d\omega'$$

If we split χ into real and imaginary parts:

$$\chi = \chi' + i\chi''$$

Then $$\chi'(\omega) + i\chi''(\omega) = \frac{-1}{\pi i} P \int_{-\infty}^{\infty} \frac{\chi'(\omega) + i\chi''(\omega)}{\omega' - \omega} d\omega'$$

Equating real and imaginary parts:

$$\boxed{\begin{aligned} \chi'(\omega) &= \frac{-1}{\pi} P \int_{-\infty}^{\infty} \frac{\chi''(\omega)}{\omega' - \omega} d\omega' \\ \chi''(\omega) &= \frac{1}{\pi} P \int_{-\infty}^{\infty} \frac{\chi'(\omega)}{\omega' - \omega} d\omega' \end{aligned}}$$

These are sometimes called the Kramers–Kronig relations, connecting the real and imaginary parts of the susceptibility. (In the language of complex-function theory, they are often called Hilbert transform relations.)

The Kramers–Kronig relations are often expressed in different, but completely equivalent forms. They are probably most often expressed in terms of the dielectric constant, which are trivially obtained by substituting

$$\varepsilon(\omega) = 1 + \chi(\omega)$$

=> $\mathrm{Re}(\varepsilon) = 1 + \chi', \mathrm{Im}(\varepsilon) = \chi''$

Also, noting again that $\chi(t)$ is real and

$$\chi(\omega) = \int_{-\infty}^{\infty} \chi(t) e^{-i\omega t} dt$$

We see that $\chi(-\omega) = \chi(\omega^*)$ (or $\chi^*(\omega^*)$, if you allow for complex ω)

Thus $\mathrm{Re}(\chi) = \chi'$ is even in ω

$\mathrm{Im}(\chi) = \chi''$ is odd in ω

This allows the Kramers-Kronig relations to be expressed entirely in terms of positive frequencies (proof left to reader)

$$\chi'(\omega) = \frac{-2}{\pi} P \int_{0}^{\infty} \frac{\omega' \chi''(\omega')}{\omega'^2 - \omega^2} d\omega'$$

$$\chi''(\omega) = \frac{2\omega}{\pi} P \int_{0}^{\infty} \frac{\chi'(\omega')}{\omega'^2 - \omega^2} d\omega'$$

Physical Consequences of Kramers-Kronig relations

1. If there is absorption, there will be dispersion, and vice versa.
2. If you can experimentally measure the spectrum of one（e.g. absorption spectrum）,the Kramers-Kronig relations may be used to obtain the other (e.g. index)
3. A sharp maximum in the absorption results in a sharp charge in sign of the index.

4. At frequencies higher than the highest resonance in the material , $\chi' < 0$,and $\chi' \to 0$ as $\omega \to \infty$.Thus

$$n^2 = \varepsilon_r = 1 + \chi' \to 1 \text{from below}$$

(This is a direct consequence easily deduced using the results developed in the homework.)

Linear Response Theory and Pulse Propagation

Now that we understand the main features of linear response, we are in a position to consider the propagation of non-monochromatic light through a medium (or a linear system in general). So far we have considered only the propagation of light at a single frequency (monochromatic light), but any we are concerned with the propagation of pulses of light.

The first thing to notice is that when we specify n at a certain frequency, we are giving the index seen by a pure harmonic wave .In other words, the phase velocity of the wave

$$E = E_0 e^{i(\omega t - \vec{k}\cdot\vec{r})}$$

$$V_p = \frac{\omega}{k(\omega)} = \frac{c}{n(\omega)}$$

This is the velocity of a wavefront (surface of constant phase)

Clearly, a pure harmonic wave extends in time (or space) to $\pm\infty$, so it is unphysical. Note that such a wave cannot carry any information.

Any realistic wave will have a finite extent in time (or space).Additionally, any wave used to transmit information (a “single”) must be modulated (in amplitude or phase or both)

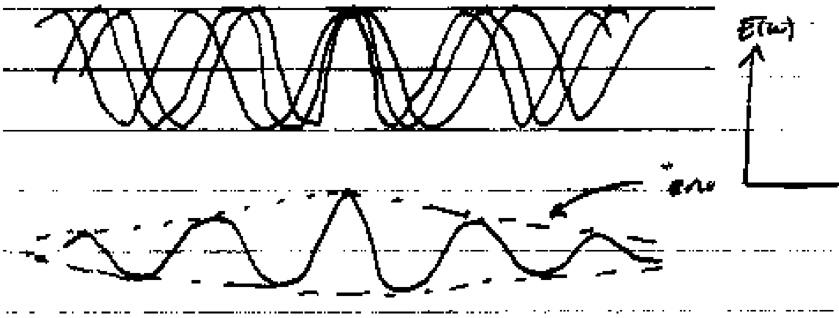
This lends us to consider the propagation of wave “groups ” or wave “packets”.

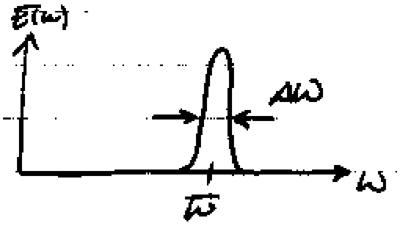
Let us construct a one-dimensional wave group (following Born+Wolf Chapter 1.3)

$$E(z,t) = \int_{\Delta\omega} E(\omega) e^{i(\omega t - kz)} d\omega$$

Where the integral runs over a narrow spectral region $\Delta\omega$, centered in some average frequency $\bar{\omega}$.

Recall the physical idea behind such a construction. One adds up many waves of nearly equal frequency, so they constructively add near the center of the packet but the fields average to zero far from the center:





Of course the sum over an infinite number of waves is just the Fourier transform integral, with $E(\omega)$ being the amplitude of the harmonic wave of frequency ω.

We will suppose that $E(\omega)$ is substantial only over a range $\underline{\Delta\omega \ll \bar{\omega}}$.This is physically equivalent to saying that there are <u>many cycles</u> in the wave group.

The spectrum of $E(\omega)$ is just a plot of Fourier amplitudes ω frequency.

There are several ways in which we might obtain the group velocity, i.e. the speed of which this wave packet propagates; here are two

(i) From the sketch at the bottom of P.132., we notice that the position of the maximum of the envelope occurs where all the waves are adding in phase. Thus the envelope will propagate with a speed determined by how fast the position propagates. In other words, at the peak of the envelope, the <u>variation</u> of <u>phase</u> with <u>frequency</u> is <u>zero</u>:

$$\phi = \omega t - kz \Rightarrow \frac{d\phi}{d\omega} = t - z\frac{dk}{d\omega} = 0$$

$$\Rightarrow t = \frac{z}{V_g} \quad \text{where} \quad \boxed{V_g = \frac{d\omega}{dk}} = \underline{\text{group velocity}}$$

(ii) From our Fourier representation

$$\int_{-\infty}^{\infty} E(\omega)e^{i(\omega t - kz)}d\omega = E(z,t) = e^{i(\bar{\omega}t - \bar{k}z)}\int E(\omega)e^{i\left[(\omega-\bar{\omega})t-(k-\bar{k})z\right]}d\omega = A(z,t)e^{i(\bar{\omega}t-\bar{k}z)}$$

$\bar{\omega}$ = “carrier frequency”

Where $A(z,t)$ is the envelope of the wave amplitudes

$$A(z,t) = \int E(\omega)e^{i\left[(\omega-\bar{\omega})t-(k-\bar{k})z\right]}d\omega$$

Note that $A(z,t)$ varies slowly with wavelength if $\Delta\omega \ll \bar{\omega}$（i.e .many cycles in pulse）.

When $\Delta\omega$ is small, we may take

$$\frac{k-\bar{k}}{\omega-\bar{\omega}}=\left.\frac{dk}{d\omega}\right|_{\bar{\omega}}$$

So $$A(z,t)=\int E(\omega)e^{i\{(\omega-\bar{\omega})[t-\left.\frac{dk}{d\omega}\right|_{\bar{\omega}}z]\}}d\omega$$

Thus on the surfaces

$$t=\left.\frac{dk}{d\omega}\right|_{\bar{\omega}}z=\frac{z}{V_g}\ ,$$

$A(z,t)$ is a <u>constant.</u>

Thus the envelope of the wave is seen to propagate at a velocity

$$\boxed{V_g=\left.\frac{d\omega}{dk}\right|_{\bar{\omega}}}$$

Note that if $n(\omega)$ =n =constant, then

$$k=\frac{n\omega}{c},V_p=\frac{n}{c}$$
$$\frac{dk}{d\omega}=\frac{n}{c},V_g=\frac{c}{n}=V_p$$

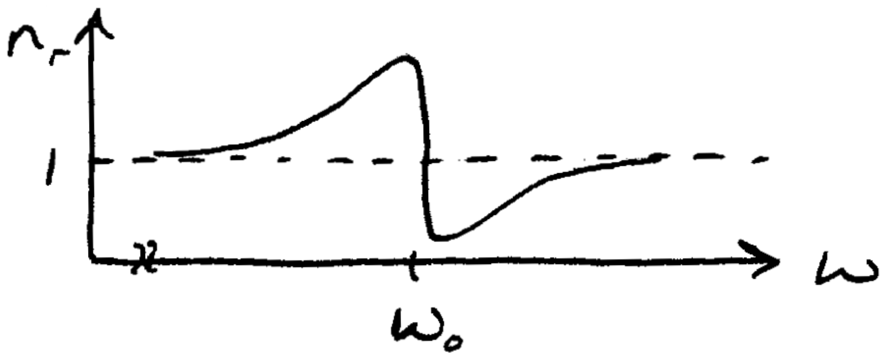
In this case the oscillations of the electric field are stationary under the envelope in a frame of reference moving with the envelope. If $n(\omega)\neq$constant

$$\frac{dk}{d\omega}=\frac{n}{c}+\frac{\omega}{c}\frac{dn}{d\omega}\Rightarrow V_g=\frac{c}{n+\omega\frac{dn}{d\omega}}\neq V_p$$

Then (in the reference frame of the envelope), the "carrier wave"appears to slip underneath the envelope (e.g. slips forward if $V_p>V_g$).

Clearly, in the case $\Delta\omega\ll\bar{\omega}$, the speed of energy or information propagation is given by the group velocity, and not the phase velocity.

Consider the real part of the index of refraction in the vicinity of an atomic resonance:



Note that since $n_r < 1$ for frequencies to the blue side of the resonance, i.e. $\omega > \omega_0$, the phase velocity exceeds the speed of light in a vacuum:

$$V_p = \frac{c}{n} > c$$

One might wonder whether this constitutes any violation of relativity. The answer is no, since a monochromatic wave extends to $\pm\infty$ in time, and thus can carry no information.

The usual answer given by most textbooks is to consider the group velocity

$$V_g = \frac{c}{n + \omega \frac{dn}{d\omega}}$$

Away from the resonance, $\frac{dn}{d\omega} > 0$, so $V_g < c$ which is clearly consisdent with relativity. If $\omega \simeq \omega_0$, i.e. it is within the absorption line, then $\frac{dn}{d\omega} < 0$ and once can have $V_g > c$. In this case, however, the signal is strongly absorbed, so one cannot advance the signal significantly in time (i,e. greater than a pulse bandwidth or so) without significant attention.

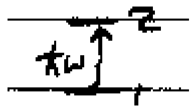
If the pulse bandwidth $\Delta\omega_p \gg \gamma$, then the spectrum overlaps regions of both positive and negative $\frac{dn}{d\omega}$.

In this case, the pulse shape strongly distorts with propagation, and it becomes impossible to define a group velocity at all, and very careful definitions of “signal velocity” are required .This problem has been definitively treated in Wave Propagation and Group Velocity in L.Bcillovin; the best we can do in over limited time is to note that no contradictions with relativity are found.

The standard textbook answer also suffers from a very serious flaw. We have noted previously that when light propagation through an atomic medium with a population inversion. (a nonclassical concept), the susceptibility is exactly the same as the classical Lorentz model susceptibility ,except that it changes sign

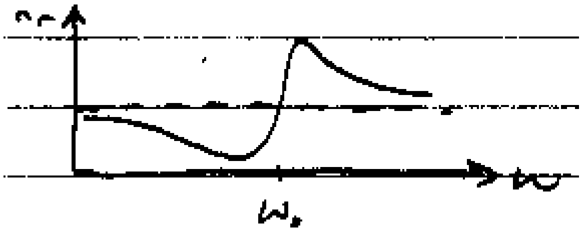
$$\chi(\omega) = N\alpha(\omega) \qquad \alpha(\omega) = \text{atomic polarizability}$$

$$\to (N_1 - N_2)\alpha(\omega)$$



Where $N_2(N_1)$ = atomic density in the upper (lower) state.

Now the refractive index looks like



Thus, over broad spectral regions away from the resonance, $\frac{dn}{d\omega} < 0$ and $V_g > c$!
This problem has been treated in detail by R.Chiao (See R.Y.Chino, Phys. Rev. A 48, R 34(993).He shows that a smooth narrow bandwidth pulse can propagate through an inverted medium with the phase, group, and energy velocities >c, without significant pulse duration. Thus the standard textbook discussion that the use of the group velocity resolves any conflict with relativity is incorrect.

Of course, despite having $V_g > c$, there is no inconsistency with relativity .The reason is that causality is not violated. The argument is simple: a population inversion simply induces a change in the sign of $\chi(\omega)$.Thus the Kramers-Kronig relations still apply, $\chi(\omega)$ remains analytic in the lower half plane ,and $\chi(\omega)$ is real with $\chi(t<0)=0$ (the latter being the crucial point)

Chiao argues that the reason superluminal group velocities are possible without violating causality is that the propagation basically induces a pulse re-shaping, and that there is no information of the peak of the pulse that is not already in the tails of the pulse (for a smooth pulse such as a Gaussian).