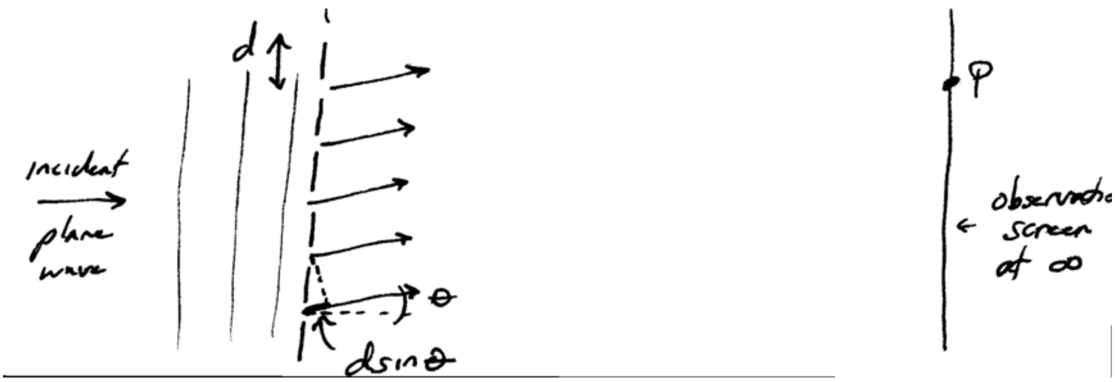
Lecture 30

Interference of N waves

We have seen that addition of 2 waves gives rise to an interference pattern of the general form $\cos\delta$ where δ is the phase difference. Since δ is a function of wavelength ($\delta = \frac{2\pi}{\lambda_0} \times OPD$), the fringe spacing is a function of wavelength, and so you might think of measuring the wavelength of light by measuring the fringe spacing. This is fine in principle, but not very accurate, since (in the presence of noise) it's not possible to determine the fringe spacing with high accuracy.

In order to increase the accuracy of the measurement, it is necessary to "sharpen up" the fringes, which is done by adding a large number of waves coherently. We will consider 2 cases: N-slit interference, and the Fabry-Perot interferometer.

Interference from N identical slits ("diffraction grating ")



We suppose, as we did with the double slit , that the screen is very far away compared with the extent of the slits (Nd), so it can be considered to be at infinity, and we need consider only parallel rays from the slits. (Alternatively, a lens could be used to focus parallel rays on the screen.)

The total field at P is just the coherent sum of all N waves:

$$E = E_0(r_1)e^{i(\omega t - kr_1)} + E_0(r_2)e^{i(\omega t - kr_2)} + \cdots + E_0(r_N)e^{i(\omega t - kr_N)}$$

$$\simeq E_0(D)e^{i\omega t}\left[e^{-ikr_1} + e^{-ikr_2} + \cdots + e^{-ikr_N}\right]$$

r_N = distance from slit N to P

↑ Maximum OPD<<distance to screen (D) => amplitudes same from all slits

$$E = E_0(D)e^{i(\omega t - kr_1)}\left[1 + e^{-ik(r_2 - r_1)} + \cdots + e^{-ik(r_N - r_1)}\right]$$

Now $r_2 = r_1$ = OPD between two adjacent slits

$$r_3 - r_1 = 2(r_2 - r_1)$$
$$r_4 - r_1 = 3(r_2 - r_1)$$
$$\vdots$$
$$r_N - r_1 = (N-1)(r_2 - r_1)$$

We know that $r_2 - r_1 = d\sin\theta \equiv \dfrac{\delta}{k} \Leftrightarrow$ phase difference between two adjacent slits.

$$E = E_0 e^{i(\omega t - kr_1)}\left[1 + e^{-i\delta} + \left(e^{-i\delta}\right)^2 + \cdots + \left(e^{-i\delta}\right)^{N-1}\right]$$

Thus we need to sum the series $1 + x + x^2 + \cdots + x^{N-1}$

This is

$$\text{sum} = \frac{x^{N-1}}{x-1}$$

$$E = E_0 e^{i(\omega t - kr_1)}\left[\frac{e^{-iN\delta} - 1}{e^{-i\delta} - 1}\right]$$

$$= E_0 e^{i(\omega t - kr_1)}\left[\frac{e^{-iN\delta/2}\left(e^{-iN\delta/2} - e^{iN\delta/2}\right)}{e^{-i\delta/2}\left(e^{-i\delta/2} - e^{i\delta/2}\right)}\right]$$

$$= E_0 e^{i(\omega t - kr_1 - (N-1)\delta/2)}\left[\frac{\sin\dfrac{N\delta}{2}}{\sin\dfrac{\delta}{2}}\right]$$

If the distance to P from the center of the grating is D, note that

$$D = r_1 + \frac{1}{2}(N-1)d\sin\theta$$

$$\rightarrow \boxed{E = E_0(D)e^{i(\omega t - kD)}\left[\frac{\sin\dfrac{N\delta}{2}}{\sin\dfrac{\delta}{2}}\right]}$$

This is of the form of a plane wave, modulated by the factor in brackets.

The intensity is as usual proportional to $|E|^2$:

$$I = I_0 = \frac{\sin^2\left(\dfrac{Nkd\sin\theta}{2}\right)}{\sin^2\left(\dfrac{kd\sin\theta}{2}\right)}$$

Or $\boxed{I = I_0 \left[\frac{\sin^2\left(N\pi \frac{d\sin\theta}{\lambda} \right)}{\sin^2\left(\pi \frac{d\sin\theta}{\lambda} \right)} \right]}$

Note that $\sin^2(N\phi)$ oscillates much more rapidly than $\sin^2\phi$.

The denominator goes to zero whenever

$$\pi \frac{d\sin\theta}{\lambda} = m\pi, m = 0, \pm 1, \pm 2, \cdots$$

⇨ $d\sin\theta = m\lambda$

What happens to I? Consider the limit $\theta \to 0$ so $\sin\theta \simeq \theta$:

$$\frac{\sin^2\left(N\pi \frac{d\sin\theta}{\lambda} \right)}{\sin^2\left(\pi \frac{d\sin\theta}{\lambda} \right)} \simeq \frac{\sin^2\left(N\pi \frac{d\theta}{\lambda} \right)}{\sin^2\left(\pi \frac{d\theta}{\lambda} \right)} \simeq \frac{\left(N\pi \frac{d\theta}{\lambda} \right)^2}{\left(\pi \frac{d\theta}{\lambda} \right)^2} = N^2$$

⇨ $\underline{\underline{I = N^2 I_0}}$ (easy to see this is true at $\frac{\delta}{2} = m\pi$ also).

For other θ, the denominator is larger => I is smaller

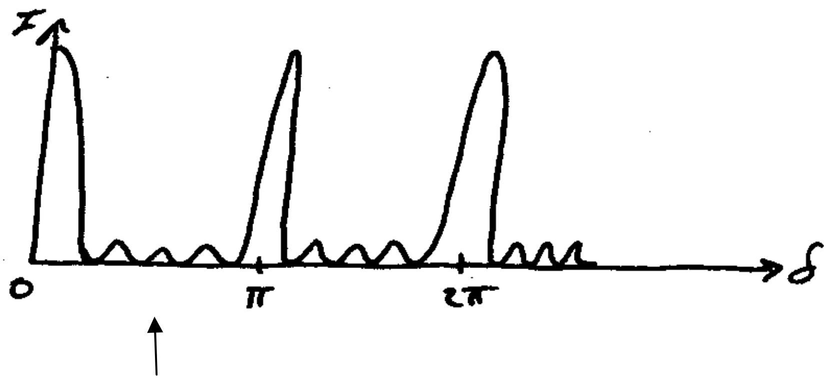
⇨ When $\boxed{d\sin\theta = m\lambda}$ we have principal maxima

↑ called the grating equation

m= order of interference (often called the diffraction order, which is a bit of misnomer!!)

Secondary maxima are obtained at $\frac{N\delta}{2} = \frac{3\pi}{2}, \frac{5\pi}{2}, \cdots$

example: N=S



N-1 minima

N-2 secondary maxima

Clearly, for N>>1, the secondary maxima are insignificant, and the principal maxima are extremely narrow. (see Guenther PP.379 ff.)

It is trivial to show (exercise for the reader) that the angular width is

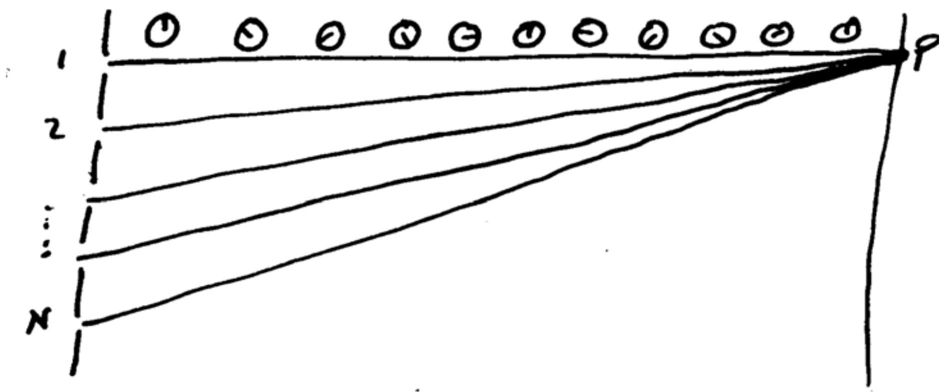
$$\Delta\theta = \frac{\lambda}{Nd\cos\theta}(= \frac{\lambda}{gratings \quad size} \ for \theta = 0!)$$

This can be made very small if the length of the illuminated grating Nd is large. Why? It's because only when every wave (i.e. all w waves) arrives at P exactly in phase do they constructively interfere. When P is slightly off the position of a principal maximum, a large number of waves are arriving with a wide spread in phases , and thus the net field is nearly zero!

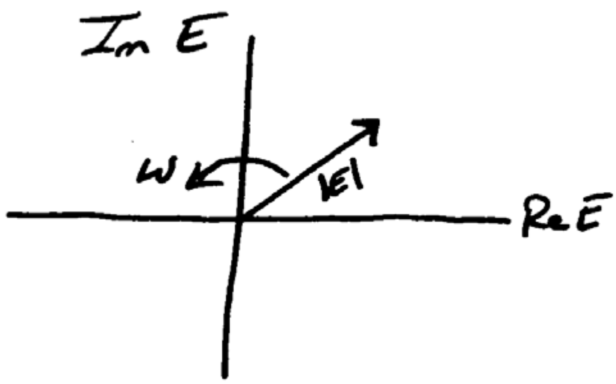
Digression: Fermat's Principle Revisited

Before concluding our discussion of interference phenomena with a discussion of coherence, I want to say just a few words about how Fermat's Principle can be thought of as a consequence of interference! We shall see that it basically arises as the result of N-wave interference, with $N \to \infty$. Our argument will follow that given by R.P.Feynman in his marvelous little book QED, chap.2 – especially ~p.50 (on reserve). The argument there is phrased as a quantum-mechanical argument, but its essence also holds classically! The only essential point is that the intensity at a point P is a sum over all possible wave amplitudes which can contribute at P (remembering their phases!!)

Begin by considering the N-slit interference problem from a phaser point of view :



Phasor



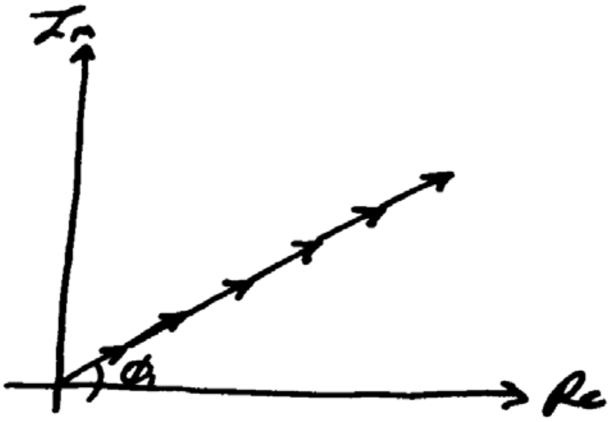
As you follow a path from any source S to P, the phasor rotates (Feynman calls it a stopwatch!) +has some final value at P.

⇨ Field from N slits is $E_{tot} = E_1 + E_2 + \cdots + E_N = |E_0|\left(e^{i\phi_1} + e^{i\phi_2} + \cdots e^{i\phi_N}\right)$

Since the amplitudes are all equal.

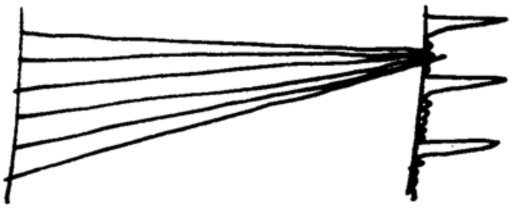
<u>Constructive</u> interference: the phases are all equal, module 2π of course.

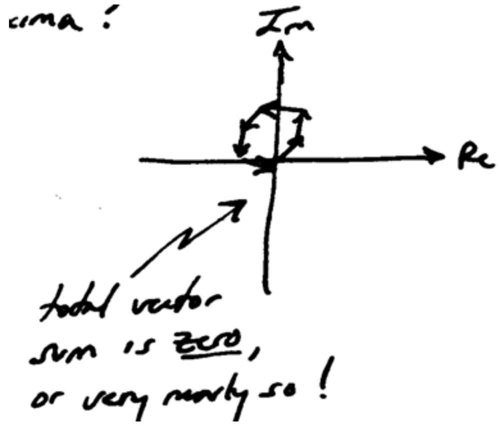
e.g



Of course, the total intensity at P is the square of the length of the sum of all the phasors.

What happens away from positions of maxima?

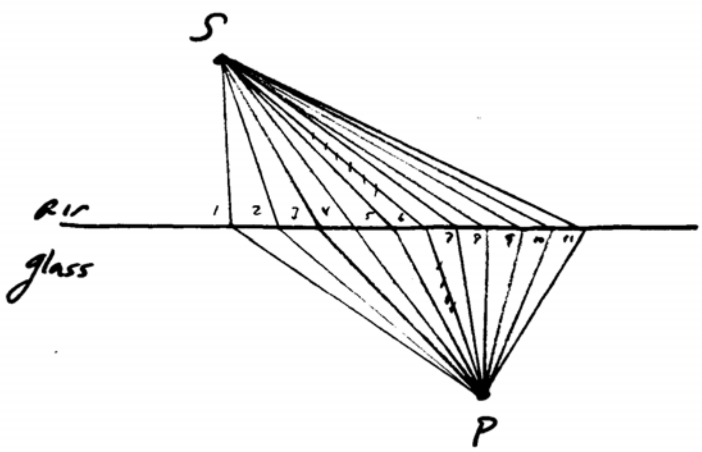




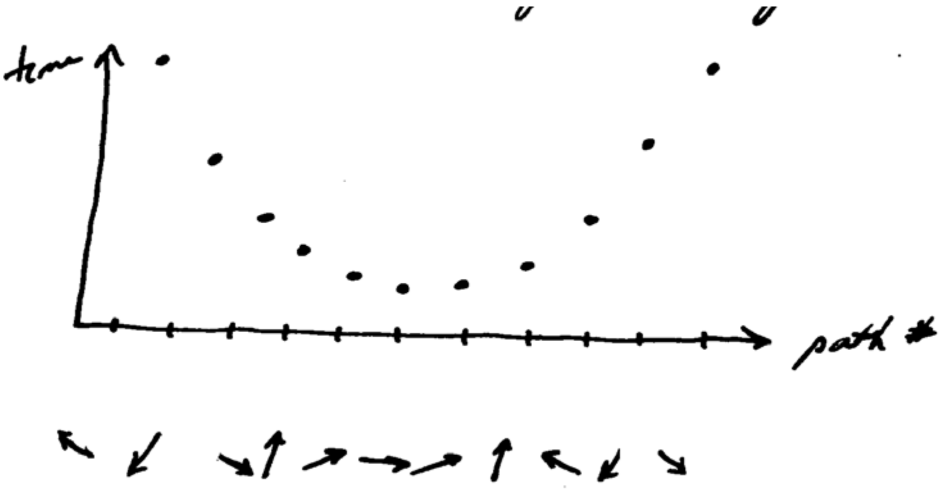
Note that as $N \to \infty$,the parks get sharper and sharper.

Now let's go back and consider how we find the path taken by light. It will suffice to consider the general question in a specific context,e.g. propagation from one medium (e.g. air) into another (e.g. glass).

There are an infinite number of paths that can be taken: let's illustrate a few:

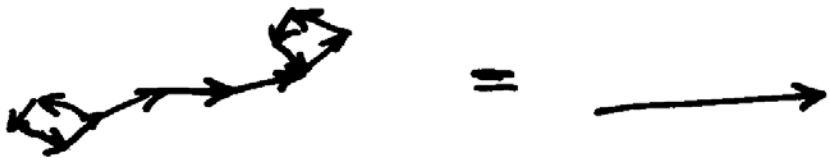


In the picture, path #6 is the one that taken the least time; the others all take longer. In fact if we plot the times, we would get something like



Corresponding phasors for each partial wave when it arrives as P .The longer delay paths give phasors that have rotated further.

Now, just as in the interference problem, add up all the phasors:



Note that the main contribution comes from the phasors that lie nearly along the same direction .This occurs where the time is least .This is what we meant by a "stationary path " in our statement of Fermat's Principle .A small change in the path near the path of least time results in only a small change in optical path length ,hence a small change in phase .It is therefore the paths near the stationary path that make the largest contribution to the intensity at P !
Note also that, away from the stationary path , small changes in path yield large changes in phase; thus the phasors just go round in circles, getting nowhere and contributing no net amplitude(see the phasors near the end points in the above fig.)

Thus, the paths away from the stationary path destructively interfere => Fermat's Principle follows!

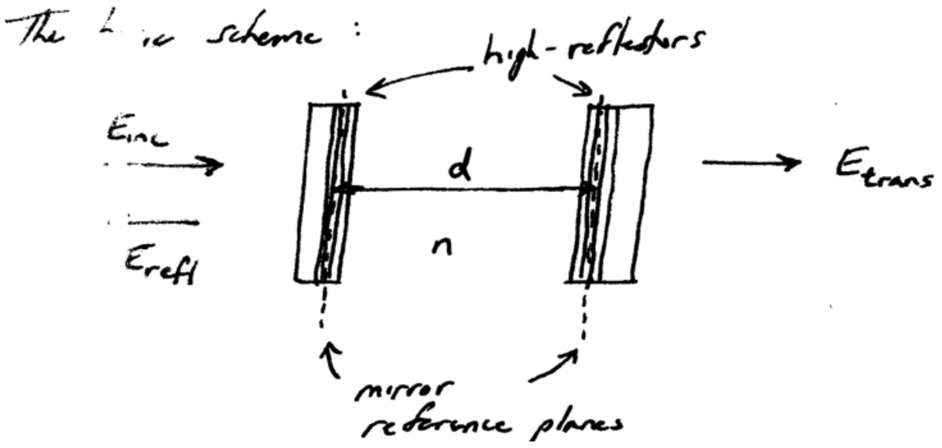
Febry-Perot Interferometer

Another way to add a large number of waves is by discussion of amplitude (rather than division of wavefront as in the N-slit problem). The most important example of this is the Febry-Perot etalon or interferometer.

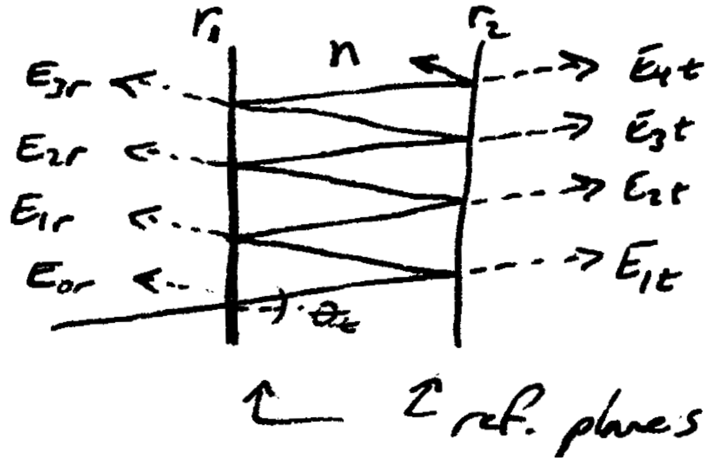
See Lipson 9.5-9.6, B+W 7.6(7^{th} ed.)

Guenther pp.106 ff. for the "standard" treatment

Siegman section 11.3 for a non-traditional but very illuminating discussion



Since the mirrors are high reflectors, we can no longer consider only one reflection at each mirror as we did in the dielectric slab problem; we must involve all the reflections.



$$t_{1,2} = \sqrt{1 - r_{1,2}^2}$$

For lossless mirrors

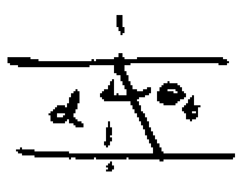
The calculation of the transmitted and reflected waves is straightforward: just add up all the "partial waves "

Incident wave = E_{inc}

Transmit through first mirror (ref. plane) $\rightarrow \; it_1 E_{inc}$

Propagate across etalon $\rightarrow it_1 e^{-i\delta_i} E_{inc}$

Where $\delta_i = \dfrac{n\omega d}{c\cos\theta_t}$



Transmit through second mirror => $E_{it} = E_{inc}\left(it_1\right)\left(it_2\right)e^{-i\delta_i} = -t_1 t_2 e^{-i\delta_i} E_{inc}$

Now to get E_{2t} ,we note that it picks up an amplitude factor $r_1 r_2$,and a phase shift $e^{-i\delta}$ .where

$\delta = \dfrac{2n\omega d}{c}\cos\theta_t$, as we described on P.253.

$$E_{2t} = \left(r_1 r_2 e^{-i\delta}\right)\left(-t_1 t_2 e^{-i\delta_i} E_{inc}\right)$$

$$E_{3t} = \left(r_1 r_2 e^{-i\delta}\right)^2 \cdot \left(-t_1 t_2 e^{-i\delta_i} E_{inc}\right)$$

$$\vdots$$

$$E_{Nt} = \left(r_1 r_2 e^{-i\delta}\right)^{N-1} \cdot \left(-t_1 t_2 e^{-i\delta_i} E_{inc}\right)$$

Since the wave picks up a factor $r_1 r_2 e^{-i\delta}$ on each round-trip through the cavity.

The total transmitted field is thus the sum

$$E_{trans} = \left[1 + r_1 r_2 e^{-i\delta} + \left(r_1 r_2 e^{-i\delta}\right)^2 + \dots\right]\left[-t_1 t_2 e^{-i\delta_1}\right] E_{inc} = \left[\sum_{l=0}^{\infty}\left(r_1 r_2 e^{-i\delta}\right)^l\right]\left(-t_1 t_2 e^{-i\delta_1}\right) E_{inc}$$

Now of course $r_1 < 1$ and $r_2 < 1$, so if we define

$$x = r_1 r_2 e^{-i\delta}$$

Then $|x| < 1$, so the infinite series converges

$$1 + x + x^2 + x^3 + \cdots = \sum_{l=0}^{\infty} x^l = \frac{1}{1-x}$$

$$E_{trans} = \frac{-t_1 t_2 e^{-i\delta_i}}{1 - r_1 r_2 e^{-i\delta}} E_{inc}$$

We can get the reflected field in exactly the same way :

$$E_{0r} = r_1 E_{inc}$$

$$E_{1r} = \left(it_1\right) r_2 \left(it_1\right) e^{-i\delta} E_{inc}$$

$$E_{2r} = r_1 r_2 e^{-i\delta} E_{1r}$$

$$\vdots$$

$$E_{Nr} = \left(r_1 r_2 e^{-i\delta}\right)^{N-1} E_{1r}$$

$$E_{refl} = \left[\sum_{l=0}^{\infty}\left(r_1 r_2 e^{-i\delta}\right)^l\right]\left(-t_1^2 r_2 e^{-i\delta}\right) E_{inc} + r_1 E_{inc}$$

$$E_{refl} = \left[r_1 - \frac{-t_1^2 r_2 e^{-i\delta}}{1 - r_1 r_2 e^{-i\delta}}\right] E_{inc}$$

Now lets' consider the transmitted intensity:

$$I_{trans} = \left|\frac{-t_1 t_2 e^{-i\delta_1}}{1 - r_1 r_2 e^{-i\delta}}\right|^2 I_{inc}$$

This is a general expression, but it will be useful to consider more specifically the special case of the Symmetrical Fabry-Perot, where $r_1 = r_2 \equiv r$, and $t_1 = t_2 \equiv t = \sqrt{1 - r^2}$.

$$I_{trans} = \frac{t^4}{\left|1 - r^2 e^{-i\delta}\right|^2} I_{inc}$$

$$\left|1 - r^2 e^{-i\delta}\right|^2 = \left(1 - Re^{-i\delta}\right)\left(1 - Re^{i\delta}\right)$$

$$= 1 - R\left(e^{-i\delta} + e^{i\delta}\right) + R^2$$

Now $= 1 - 2R\cos\delta + R^2$ $\qquad R = r^2$

$$= 1 - 2R(1 - 2\sin^2\frac{\delta}{2}) + R^2$$

$$= \left(1 - R\right)^2 + 4R\sin^2\frac{\delta}{2}$$