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Abstract: We discover numerically that a moving wave packet in a chaotic billiard will always evolve into a quantum state, whose density probability distribution is exponential. This exponential distribution is found to be universal for quantum chaotic systems with rigorous proof. In contrast, for the corresponding classical system, the distribution is Gaussian. We find that the quantum exponential distribution can smoothly change to the classical Gaussian distribution with coarse graining. This universal dynamical behavior can be observed experimentally with Bose-Einstein condensates.



Quantum "random" gas with exponential density distribution

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Universal behavior in quantum chaotic dynamics

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1. Introduction

The experimental realization of ultra-cold atoms has provided fresh perspectives and new opportunities for many traditional fields of physics [1–3], for example, Bloch oscillations [4, 5], BCS-BEC crossover [6, 7], and quantum turbulence [8, 9]. In particular, it was pointed out recently that it is now possible to study quantum chaos with Bose-Einstein condensates [10]. In the past, intensive study of quantum chaos has yielded many important results, elucidating the intriguing correspondence between quantum mechanics and classical mechanics [11–15]. They include the Wigner distribution of energy level spacing [16], the "scarring" of eigenstates along the classical periodic orbits [17,18], Schnirelman's ergodic theorem on eigenstates [19, 20]. However, all of these are the static properties of a quantum chaotic system, concerning only its eigenval-

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ues and eigenstates. In contrast, the quantum chaotic dynamics has received far less attention [21]. With the Bose-Einstein condensate and the technique of CCD imaging, we can now not only study experimentally the static properties of a quantum chaotic system [22] but also study its dynamics [10].

Motivated by this new possibility, we investigate the dynamics in a quantum chaotic system. By following a pioneering work [21], we study the time evolution of a wave packet inside a ripple quantum billiard [23].

We find numerically that the wave packet evolution always leads to an "equilibrium" state, where the density probability is exponentially distributed. With rigorous proof, we show that this exponential distribution is universal for all quantum chaotic systems. This exponential law is in stark contrast with the evolution of a "cloud" of classical particles in the same billiard, which always





Figure 1 (online color at www.lphys.org) Dynamical evolution of a moving wave packet in a ripple billiard (a0)–(a4) and a square billiard (b1)–(b4). (c) – the time evolution of the probability for the density being above the average. The red-dashed line is for 1/e with *e* being the Euler's number. (d) – the density probability distribution functions at $t = 12T_s$. The red solid line is for $\exp(-n_0)$. The inset shows the semilog plot of the same set of data. $\delta n = 0.02$ is used to obtain the results in (d)

leads to a Gaussian distribution. Furthermore, we demonstrate that the quantum-classical transition between the exponential distribution and the Gaussian distribution can be achieved with coarse graining. Now this exponential distribution has been observed experimentally with a BEC in one-dimensional optical lattice [24].

2. Dynamical evolution in a quantum chaotic billiard

We consider the following Schrödinger equation with the units $\hbar = 2m=1$

$$i\frac{\partial\Psi}{\partial t} = \left(-\frac{\partial^2}{\partial x^2} - \frac{\partial^2}{\partial y^2} + V_b\right)\Psi,\tag{1}$$

where V_b represents the hard-wall potential for a ripple billiard (see Fig. 1a0). The ripple billiards have the advantage of having an exact Hamiltonian matrix in terms of elementary functions [23]. The left and right walls are described by functions $\mp [b - a\cos(2\pi y/L)]$, respectively. The initial wave function is a moving Gaussian wave packet $\Psi(x, y, t = 0) = \Psi_G(x, y) \exp(iv_0 x/2)$, where $\Psi_G(x, y)$ is the ground-state wave function in a harmonic trap $V_h = (\omega_x^2 x^2 + \omega_y^2 y^2)/4$.

The time evolution of the wave packet density $n(x, y, t) = |\Psi(x, y, t)|^2$ is shown in Fig. 1a1–Fig. 1a4 for the parameters $\omega_x = \omega_y = 2$, $v_0 = 10$, L = 30, b = 15, and a = 6. The time is in the unit of $T_s = 2(a + b)/v_0$, which is the longest time to make a round-trip along the x direction inside the billiard. All the parameters are chosen from relevant BEC experiments [1]. As clearly seen in the figure, the wave packet moves and expands, and eventu-

ally gets reflected multiple times by the curved hard walls. As a result, the wave packet gets smeared out in the billiard. After about $t = 6T_s$, the density of the wave packet reaches an "equilibrium" state, where the overall feature of the wave packet no longer changes. For comparison, the evolution of the same wave packet in a square billiard (a=0) is shown in Fig. 1b1 – Fig. 1b4. The difference is obvious: in the square billiard, the wave packet always has certain patterns, looking very regular, and it never settles into any fixed pattern.

To describe quantitatively the contrasting images between Fig. 1a3 – Fig. 1a4 and Fig. 1b3 – Fig. 1b4, we introduce the density probability distribution function P(n, t), which is the probability of the wave packet density being between $n - \delta n/2$ and $n + \delta n/2$,

$$P(n,t) = \frac{S\left(n - \delta n/2, t\right) - S\left(n + \delta n/2, t\right)}{\delta n S_{\text{total}}},$$
(2)

where S_{total} is the area of the whole billiard and S(n, t) is the area of the regions where the density is larger than n. Since the wave packet in our calculation is normalized to one, the averaged density is $n_s = 1/S_{\text{total}}$. With the averaged density, we define a dimensionless density $n_0 = n/n_s$. The dimensionless probability distribution function is then $P_0(n_0, t_0) = n_s P(n, t)$ with $t_0 = t/T_s$.

shows time evolution the Fig. 1cof $s_a(t) = S(n_s, t)/S_{\text{total}}$, the probability of having density above the average. For the square billiard (a=0), $s_a(t)$ exhibits large-amplitude oscillations, reflecting the ever-changing regular patterns in the density. Very differently and also strikingly, for the ripple billiard, s_a quickly reaches a plateau, fluctuating slightly around 1/e $(e \approx 2.718$ is the Euler's number). This signals that the wave packet eventually evolves into an "equilibrium" state, where the overall feature of the wave function no longer changes. The distribution function $P_0(n_0, t_0)$ also has similar behavior: for the square billiard, P_0 always changes with time; in contrast, for the ripple billiard, P_0 settles into an "equilibrium" function. As shown in Fig. 1d, we find by numerical fitting that the "equilibrium" function is in fact exponential,

$$P_0^{\rm eq}(n_0) = \exp(-n_0)\,. \tag{3}$$

This simple exponential law is also found numerically for the widely studied stadium billiard [17,21]. The evolution of a wave packet in Fig. 1 can also be regarded as an evolution of a BEC without interaction. After adding interaction and solving the Gross-Pitaevskii equation, we find that the density distribution of a BEC can also reach the exponential distribution after long-time evolution. These numerical calculations indicate that the exponential law in Eq. (3) is likely universal and may apply in any quantum chaotic systems and in many different settings.

In Fig. 2a, we have plotted the momentum distribution of the wave function shown in Fig. 1a4. The momentum distribution has a ring-shaped structure with a



Figure 2 (online color at www.lphys.org) (a) – density distribution in the momentum space for the quantum state shown in Fig. 1a4. (b) – probability distribution for the momentum distribution in (a). In obtaining (b), the momentum space is divided into concentric rings with equal spacing $\delta k = \pi/30$ and each ring is then divided into equal pieces so that each piece has the area of $\sim \delta k \times \delta k$. The density at each unit piece is renormalized with the averaged density of the ring where the unit piece belongs. The large fluctuations in (b) are caused by the small number of sample points

spotty look. To understand this structure, let us consider the time evolution of a "cloud" of non-interacting identical classical particles, whose initial velocity distributions are identical to the wave packet in Fig. 1a0, that is, $G(\mathbf{v}, t = 0) \sim \exp(-|\mathbf{v}-\mathbf{v}_0|^2/2\sigma_v^2)$. For a chaotic billiard with hard-wall boundary (e.g., the ripple billiard), after long-time evolution, the velocity distribution will become $G(\mathbf{v}, t \gg T_s) \sim \exp[-(|\mathbf{v}|^2 - v_0^2)/2\sigma_v^2]$, which is exactly a ring-shaped distribution with a maximum at $v = v_0$. Since the classical distribution $G(\mathbf{v}, t \gg T_s)$ is very smooth, the spotty look in Fig. 2a has a quantum origin. As we have done for the density in the real space, we use similar statistics to quantify the spotty or random image in Fig. 2a. We first divide the whole momentum space into a series of rings with equal small spacing and then cut each ring into small pieces of the same area. In this way, we obtain a set of momentum densities. This set of densities does not follow the exponential law. However, after renormalizing the densities of each ring with its averaged density, we again find an exponential distribution as shown in Fig. 2b.

Moreover, following [21], we construct a semiclassical wave function with the momentum distribution G

$$\Psi(x, y, t \gg T_s) \propto \int dv_x dv_y G(\mathbf{v}, t \gg T_s) \times$$

$$\times \exp\left\{-i \left[v_x x/2 + v_y y/2 + \varphi_r(v_x, v_y, t) \right] \right\},$$
(4)

where $\varphi_r(v_x, v_y, t)$ is the random phase caused by the classical chaos. The density of this wave function is shown in Fig. 3a. Although it appears different in structure from Fig. 1a4, the random wave function Ψ in Eq. (4) also obeys the exponential law (black solid line in Fig. 3b). Due to the large sampling size in this case, the numerical result fits the exponential function $\exp(-n_0)$ almost perfectly.

All the above results point to the universality of the exponential distribution. This will become apparent as we give a rigorous proof in the following. We divide the region of the dynamical evolution into N equal parts. Then the quantum state $|\Psi(t)\rangle$ can be approximated as

$$|\Psi(t)\rangle \approx \sum_{j=1}^{N} \alpha_j(t) |x_j, y_j\rangle.$$
 (5)

Here

$$\alpha_j(t) = \frac{\int\limits_{\Xi_j} \Psi(x, y, t) dx dy}{\sqrt{S_{\text{total}}/N}}$$

with Ξ_j denoting the *j*th part. Obviously, α_j 's are complex numbers, satisfying the normalization condition

$$\sum_{j=1}^{N} |\alpha_j|^2 = 1.$$
 (6)

We assume that for a fully chaotic classical system, its corresponding quantum dynamics will always drive the system to states, where the α_j 's are a set of random complex numbers that satisfy the above normalization condition. With these considerations, the probability of $|\alpha_j|^2$ being between γ_j and $\gamma_j + d\gamma_j$ is

 $P(\gamma_j)d\gamma_j = \tag{7}$





Figure 3 (online color at www.lphys.org) (a) – density of the randomly superposed plane waves. It is computed according to Eq. (4) with 10,000 plane waves for an area of 400×400 . (b) – density probability distributions for the random quantum state in (a). The distribution functions are obtained by dividing this whole area equally into different square cells, 0.1×0.1 (black line), 2×2 (green line), and 8×8 (red line)

$$=\frac{\int d^2 \alpha_1 \dots d^2 \alpha_N \,\delta\big(\gamma_j - |\alpha_j|^2\big) \,\delta\bigg(1 - \sum_{i=1}^N |\alpha_i|^2\bigg)}{\int d^2 \alpha_1 \dots d^2 \alpha_N \,\delta\bigg(1 - \sum_{i=1}^N |\alpha_i|^2\bigg)} d\gamma_j \,.$$

It is easy to find that

=

$$\int_{0}^{1} P(\gamma_j) d\gamma_j = 1$$

After straightforward calculations [25, 26], we have

)
$$P(\gamma_j) = (N-1)(1-\gamma_j)^{N-2}$$
. (8)



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Figure 4 (online color at www.lphys.org) The 959th eigenstate of the ripple billiard and its density probability distribution. The parameters for the ripple billiard are the same as in Fig. 1. The inset in (b) is the semilog plot of the same data

In the limit of $N \to \infty$, we have

$$\alpha_j(t) \approx \frac{\Psi(x_j, y_j, t)}{\sqrt{n_s N}}$$

and the density probability distribution for the *j*th part becomes $P(n_j) = \exp(-n_j)$, where $n_j = |\Psi(x_j, y_j)|^2/n_s$. Because different part satisfies the same exponential probability distribution, the dimensionless density probability distribution $P_0(n_0)$ defined for the whole region satisfies the exponential probability distribution $\exp(-n_0)$.

The above rigorous proof indicates that this exponential law is universal and should appear in any quantum chaotic systems and many different settings in the momentum space. We notice that this exponential law is known to exist in the speckle pattern of light [27]. However, it seems that this exponential law has been limited to this special system, and has never been fully discussed in the context of quantum chaos or explored for broad applications. In fact, the density probability distribution of an eigenstate was studied in literature in terms of amplitude probability distribution; however, no similar universal behavior was found [28]. We have checked the eigenstates of the ripple billiards numerically and found that this is indeed the case. One example is shown in Fig. 4, where the density probability distribution is compared to both the exponential distribution and the Porter-Thomas distribution [29]. It is clear that it does not fit both of the distributions although it fits better with the Porter-Thomas distribution.

3. Quantum-classical correspondence

The established exponential law is in fact a quantum phenomenon. We can imagine to throw randomly many, many classical particles into the billiard. By the central limit theorem, the resulted density probability distribution is a Gaussian with its peak located at the averaged density, drastically different from the exponential distribution for a quantum gas. To confirm this, we have simulated the evolution of a cloud of non-interacting classical particles with the same spatial and velocity distributions as the wave packet studied in Fig. 1. The results are shown in Fig. 5, where we see the classical cloud expands and reflects very much like its quantum wave packet in Fig. 1. However, the density probability distribution function for the classical cloud after long time evolution is Gaussian (see Fig. 5b) as we just argued. This shows that the exponential distribution is a quantum effect and can be used in experiment to tell whether a highly-excited gas is quantum or classical (thermal).

The central question in the study of quantum chaos is how the classicality observed in our daily life emerges from the underlying quantum world [30]. Therefore, it is very interesting to see how the classical Gaussian distribution emerges from the quantum exponential distribution. We find that the quantum-classical correspondence in this regard is built by coarse graining. We have computed the density of the random wave in Eq. (4) for an area of 400×400 . To obtain the density probability distribution P_0 , we then divide the whole area equally into small square cells and sample the averaged densities in these small cells. We find that the distribution function changes its shape with the size of the cell: as the cell size increases, the distribution changes from exponential to Gaussian as shown in Fig. 3b. The same transition is observed for the density in Fig. 1a4 with coarse graining although the fluctuations are large due to the small sampling size. Besides its fundamental significance, this coarse graining result has also implications in practice: with a camera of low resolution, one is likely to observe a Gaussian distribution even for a quantum gas.





Figure 5 (online color at www.lphys.org) Evolution of a cloud of classical particles. (a0) – (a4) are plotted with 20,000 particles while the statistics shown in (b) is done with 1 million particles. The cloud initially has the exact same spatial and velocity distribution as the initial wave packet in Fig. 1. The unit cell used for obtaining the statistics in (b) is 0.6×0.6

4. Discussion

We have been calling the quantum state reached after longtime evolution "equilibrium" state. This is because we can not resist the temptation to compare it with the equilibrium state in thermodynamics. In thermodynamics, a system will always evolve into an equilibrium state after fully relaxed. After the equilibrium is reached, the state still changes microscopically; however, the statistical distribution function no longer changes. It is similar here: after a certain long time ($\sim 6T_s$ in our case), the quantum state still changes locally while its overall structure remains the same and the density probability distribution function sticks to the exponential law. This reaching of an "equilibrium state" should also be closely related to or provide a fresh perspective to the quantum ergodicity, which has been discussed extensively in terms of eigenfunctions [19, 20].

It is worthwhile to compare the hyper-sphere in the Hilbert space defined by the normalization condition (6) to the hyper-sphere in the phase space defined by a constant energy. As we know, for almost all the microscopic states in the phase space sphere, they share the same macroscopic characteristics, such as the same pressure and the same thermal expansion coefficient. Similarly, with the understanding that we have gained so far, almost all the states in the Hilbert space should have the exponential distribution. It is true that there are many states such as the eigenstate shown in Fig. 4, which do not have the exponential distribution. However, we expect the measure of these states is zero. In fact, the rigorous proof and the numerical simulation provided above seem to indicate that the dynamical evolution in the hyper-sphere in the Hilbert space is ergodic. In light of this discussion, it is likely that this exponential distribution is related to the thermalization of isolated quantum systems [31–33].

Quantum chaotic system has been defined as the quantum system whose corresponding classical system is chaotic. There have always been efforts to define quantum chaos without referring to classical dynamics [20]. With the exponential distribution law, we may be now ready to do this. One might choose to define a quantum chaotic system as a quantum system which has the ability to drive an initially regular wave packet to a coherent superposition of completely random wave functions, which has the exponential density probability distribution. We finally note that for the exponential distribution, the density fluctuation is $\delta n^2 = n^2$, which is much larger than the Gaussian distribution. In other words, the quantum fluctuation in density is much larger than the classical fluctuation. This large quantum fluctuation in density may help to explain the formation of stars if a wave function for the whole universe is assumed [34, 35].

Besides its theoretical significance, our result has potential applications in the field of ultracold atoms, where the study of non-equilibrium ultracold atoms is a hot topic [8, 9, 33, 36, 37]. A cloud of atoms can be prepared in their lowest energy state and then be transferred to a nonequilibrium state by some external fields (chaotic billiard is one example) [38]. The action of these external fields may or may not destroy the quantum nature of the atomic gas. The question is then: how one can distinguish experimentally whether the gas is a quantum gas or classical gas. The most straightforward way is to measure the distribution of density in the cloud. Our result shows that for a chaotic system, the density distribution of a quantum gas is drastically different from that of a classical gas. The former is exponential while the latter is Gaussian.

The exponential density distribution of a quantum gas was just observed experimentally with a BEC in one dimensional lattice and used to reveal quantum criticality [24].

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