Lecture 40

Now if the field is stationary, so that the autocorrelation is independent of time origin, it must be of the form

$$\left\langle E^*(t)E(t')\right\rangle = \Gamma(t'-t)$$

(i.e. it depends only on the time difference $\tau = t'-t$)

$$\left\langle E^*(\omega)E(\omega')\right\rangle = \int_{-\infty}^{\infty} dt e^{-i(\omega'-\omega)t} \int_{-\infty}^{\infty} dt'\Gamma(t'-t)e^{-i\omega'(t'-t)}$$

（where we used $\omega't' - \omega t = \omega'(t'-t) + \omega't - \omega t = \omega'(t'-t) + (\omega'-\omega)t$ ）

Now $\tau = t'-t \Rightarrow dt' = d\tau$

$$\left\langle E^*(\omega)E(\omega')\right\rangle = \int_{-\infty}^{\infty} dt e^{-i(\omega'-\omega)t} \int_{-\infty}^{\infty} dt'\Gamma(\tau)e^{-i\omega'\tau} = \delta(\omega'-\omega)\int_{-\infty}^{\infty} d\tau\Gamma(\tau)e^{-i\omega'\tau}$$

Thus we have

$$\boxed{S(\omega) = \left\langle E^*(\omega)E(\omega)\right\rangle = \int_{-\infty}^{\infty} d\tau\Gamma(\tau)e^{-i\omega\tau}}$$ Weiner-Khinchine theorem

Thus measurement of the field autocorrelation, which is what the Michelson fringe pattern gives directly, yields the power spectrum by Fourier transform. Conversely, if you know the spectrum, then the field autocorrelation is determined.

Physical significance of $\gamma(\tau)$:

$$I = \frac{I_0}{2}\left[1 + \mathrm{Re}\,\tilde{\gamma}(\tau)\right] = \frac{I_0}{2}\left[1 + \gamma(\tau)\right]$$

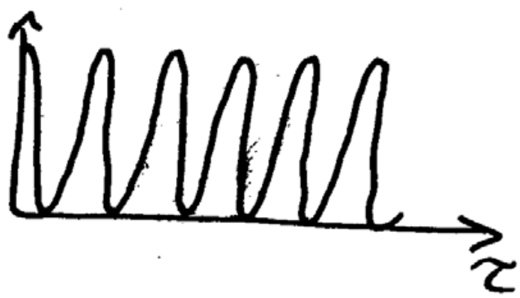
(i) for a perfectly coherent wave $\delta(\omega) = \delta(\omega - \omega_0) \Rightarrow \gamma(\tau) = \cos\omega_0\tau$

$$I_{\max} = \frac{I_0}{2}[1+1] = I_0$$

$$I_{\min} = \frac{I_0}{2}[1-1] = 0$$

Fringe visibility

$$V \equiv \frac{I_{\max} - I_{\min}}{I_{\max} + I_{\min}} = \frac{I_0}{I_0} = 1$$



（i） More generally, $\gamma(\tau)$ is of the form $\left|\gamma(\tau)\right|\cos\omega_0\tau$,i.e. a damped oscillatory function

$$I_{max} = \frac{I_0}{2}\left[1+|\gamma(\tau)|\right]$$

$$I_{min} = \frac{I_0}{2}\left[1-|\gamma(\tau)|\right]$$

$$\Rightarrow V = \frac{I_0\left[1+|\gamma(\tau)|\right]-I_0\left[1-|\gamma(\tau)|\right]}{I_0\left[1+|\gamma|\right]+I_0\left[1-|\gamma|\right]}$$

$$\boxed{V = |\gamma(\tau)|}$$ The <u>coherence function</u> is just the <u>fringe visibility</u>!

$|\gamma(\tau)| = 1 \Rightarrow$ complete coherence

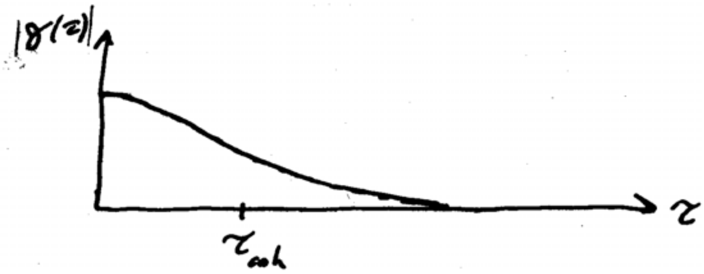
$|\gamma(\tau)| = 0 \Rightarrow$ complete incoherence

$0 < |\gamma(\tau)| < 1 \Rightarrow$ <u>partial coherence</u>

$$\tilde{\gamma}(\tau) = \frac{\langle E^*(t)E(t+\tau)\rangle}{\langle |E|^2\rangle}$$

An autocorrelation (suitably time-averaged) is also intuitively just a measure of how well you can predict the value of the field at one time $t+\tau$ given the field at t. Complete coherence means you can predict it with certainty. <u>Partial</u> coherence means that there is <u>some</u> <u>correlation</u> between the fields at two different times, but the correlation is not perfect; there may have born some phase shifts or amplitude fluctuations in between.

The time over which a field is strongly coherent with itself, i.e. where the autocorrelation has significant amplitude, is called the <u>coherence time</u>.



The quantity $l_c = c\tau_{coh}$ is often called the (longitudinal) <u>coherence length</u>.

The <u>quantitative</u> characterization of the coherence time is a little bit arbitrary .There are <u>two principal</u> ways of defining the coherence time in the literature.

(i) Given $\Gamma(\tau) = \langle E^*(t)E(t+\tau)\rangle$

The coherence time may be defined as the <u>root mean square</u> <u>width</u> of $|\Gamma|^2$:

$$\tau_c^2 = \frac{\int_{-\infty}^{\infty} \tau^2 |\gamma(\tau)|^2 d\tau}{\int_{-\infty}^{\infty} |\gamma(\tau)|^2 d\tau}$$

Similarly, the effective spectral width of the light may be defined as

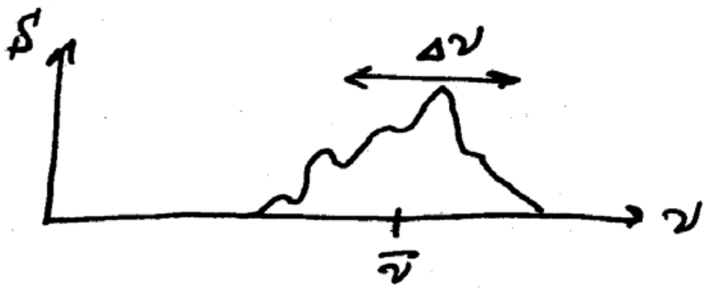
$$(\Delta\upsilon)^2 = \frac{\int_0^\infty (\upsilon-\bar{\upsilon})^2 S^2(\upsilon)\,d\upsilon}{\int_0^\infty S(\upsilon)^2\,d\upsilon} \qquad S(\upsilon) = \int_{-\infty}^{\infty} \Gamma(\tau)\mathrm{e}^{-2\pi i\upsilon\tau}d\tau$$

Where

$$\bar{\upsilon} = \frac{\int_0^\infty \upsilon S^2(\upsilon)\,d\upsilon}{\int_0^\infty S(\upsilon)^2\,d\upsilon} \quad = \text{mean frequency}$$

And $S(\upsilon)$ is the spectral density vs. freq. $\upsilon = \frac{\omega}{2\pi}$.

e.g.



It is possible to show using standard theorems on Fourier transform relations (see Mandel +Wolf P4.3.3) that these satisfy $\tau_c \cdot \Delta\upsilon \geq \frac{1}{4\pi}$.

Equality is obtained only if the spectrum is a Gaussian. This definition of the coherence time is useful generally when the light is quasi-monochromatic and the spectrum has a "reasonably well-defined peak"

(ii) Another common definition is to use the normalized degree of coherence

$$\tilde{\gamma}(\tau) = \frac{\Gamma(\tau)}{\Gamma(0)} = \frac{\langle E^*(t)E(t+\tau)\rangle}{\langle |E|^2 \rangle}$$

And define the coherence time as

$$\tau_c = \int_{-\infty}^{\infty} |\tilde{\gamma}(\tau)|^2\,d\tau$$

It turns out (see Mandel +Wolf 4.3.3 for proof) that the width can be written as

$$\Delta\upsilon = \frac{|\Gamma(0)|^2}{\int_0^\infty S(\upsilon)^2\,d\upsilon}.$$

With these definitions $\tau_c \cdot \Delta\upsilon = 1$ always.

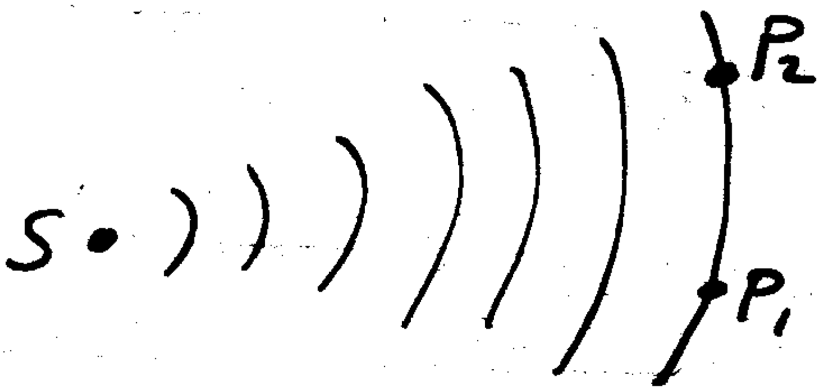
It should be noted that the above two definitions of the coherence time give roughly similar numbers for quasi-monochromatic light, but can give significantly different result for complicated, broad-band light .

Spatial Coherence

The idea of spatial coherence of a light wave is closely analogous to the concept of temporal coherence. The fundamental issue is: at a frozen instant in time, given a field at one point in space, how well can you predict what it will be in another point in space?

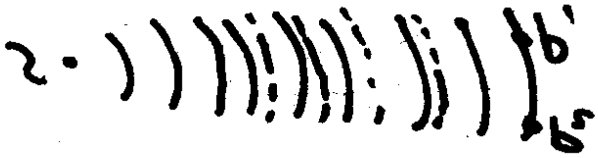
Example: complete spatial coherence

- Consider emission from a point source (which may emit light with a randomly fluctuating field!)



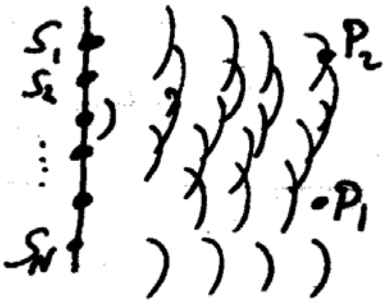
Clearly, P_1 and P_2 are on the same wavefront, so there is complete spatial coherence between these two points.

- Even if S is temporally incoherent :



Example: complete spatial incoherence

- Consider emission from an extended source of dipoles which oscillate ## random phases and/or amplitudes (uncorrelated dipoles)
- Look at the wave near the source



Clearly there is no correlation between the waves at P_1 and P_2 => complete incoherence.

Note that, just as we did for the case of temporal coherence, we phrase the question of spatial coherence in terms of the preserve(or observe) of correlations between fields, which can be quantified by the means of correlation functions.

We thus extend the definition of Γ to:

$$\Gamma(\vec{r}_1, \vec{r}_2, \tau) = \left\langle E^*(\vec{r}_1, t) E(\vec{r}_2, t+\tau) \right\rangle$$

(again, for stationary fields this is independent of t.)

$\Gamma(\vec{r}_1, \vec{r}_2, \tau) =$ mutual coherence function.

Similarly, we can extend $\tilde{\gamma}(z)$ to

$$\tilde{\gamma}(\vec{r}_1, \vec{r}_2, \tau) = \frac{\Gamma(\vec{r}_1, \vec{r}_2, \tau)}{\sqrt{|\Gamma(\vec{r}_1, \vec{r}_1, 0)||\Gamma(\vec{r}_2, \vec{r}_2, 0)|}}$$ ="complex degree of coherence "

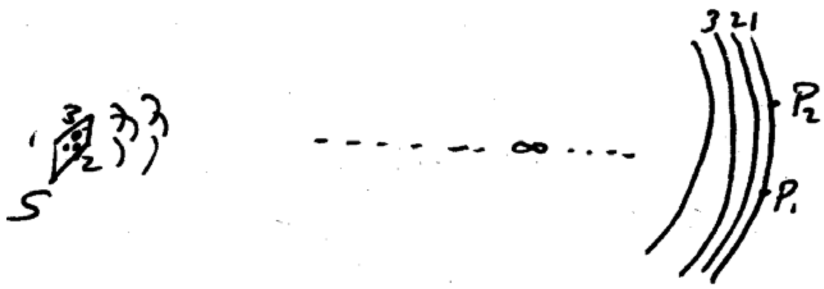
Clearly, temporal and spatial coherence are connected by propagation, but we can consider spatial coherence to be measured by

$\Gamma(\vec{r}_1, \vec{r}_2, 0)$ or $\tilde{\gamma}(\vec{r}_1, \vec{r}_2, 0)$.

Note that in general $0 \le |\tilde{\gamma}| \ll 1$, 0= complete incoherence, and 1= complete coherence.

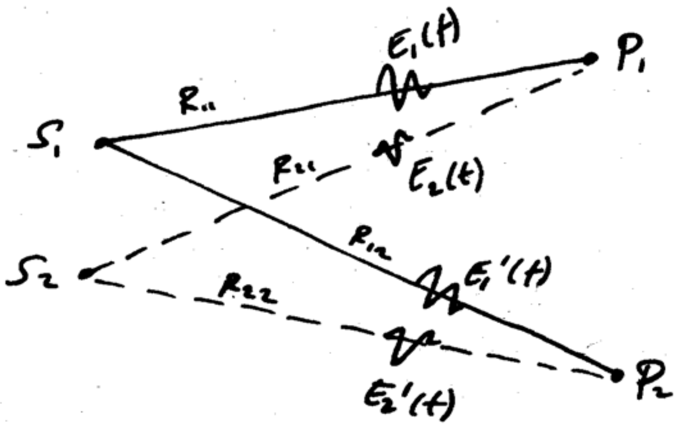
Now, it might seem, given our picture on the previous page, that spatial coherence is always negligible when the source is extended and consists of a large number of uncorrelated dipole emitters. Life turns out to be more interesting than that, however.
Surprise: the spatial coherence of light increases with propagation.

A very naïve argument can make this statement at least sound plausible: " at a large enough distance, any source looks like a point source!"



At P_1 and P_2, the multiple sources clearly give rise to temporal incoherence, but since all the waves look like spherical waves centered on S (which looks negligibly small), the wave has acquired spatial coherence!

A somewhat more sophisticated argument is given in Mandel +Wolf P 4.2.2. Consider the emission from two uncorrelated point sources S_1 and S_2, and look at the net field at two observation points P_1 and P_2.



If $|R_{11} - R_{12}| < c\tau_c$, then

$$E_1'(t) = E_1(t)e^{i\phi_1} \quad (\phi_1 = 0 \text{ if they are on the same wave front})$$

Similarly $|R_{21} - R_{22}| < c\tau_c \Rightarrow E_2'(t) = E_2(t)e^{i\phi_2}$

(the phases ϕ are just fixed by geometry – the positions of P_1 and P_2 - and are not fluctuating variables).($\phi_1 = |R_{11} - R_{12}| \cdot k$)

Now
$$E(P_1) = E_1(t) + E_2(t)$$
$$E(P_2) = E_1'(t) + E_2'(t) = E_1(t)e^{i\phi_1} + E_2(t)e^{i\phi_2}$$

S_1 and S_2 uncorrelated => E_1 and E_2 uncorrelated

$$\langle E_1^*(t) E_2(t) \rangle = 0$$

However, the <u>total</u> fields at P_1 and P_2 are correlated, since the <u>sum</u> of the two waves at each point <u>looks nearly the same</u>.

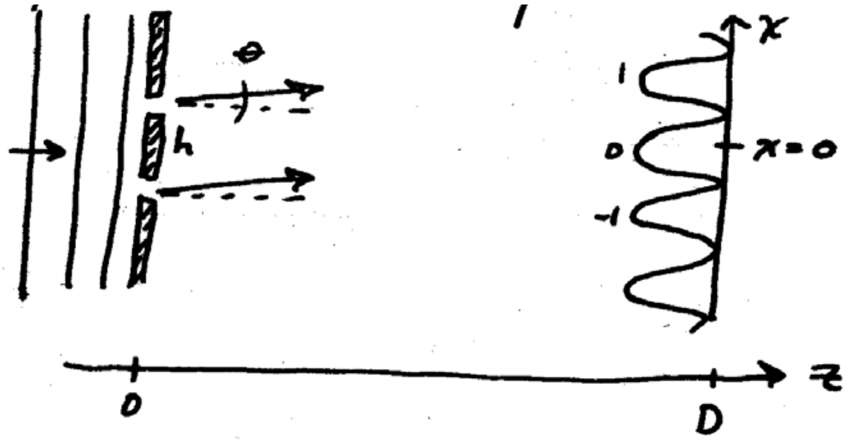
$$\Gamma(P_1, P_2, 0) = \left\langle \left[E_1^* + E_2^*\right]\left[E_1 e^{i\phi_1} + E_2 e^{i\phi_2}\right] \right\rangle$$
$$= \left\langle E_1^* E_1 e^{i\phi_1} \right\rangle + \left\langle E_2^* E_2 e^{i\phi_2} \right\rangle \quad \longleftarrow \underline{\text{NOT}} = 0!$$
$$+ \left\langle E_2^* E_1 e^{i\phi_2} \right\rangle + \left\langle E_1^* E_2 e^{i\phi_2} \right\rangle$$
$$\quad\quad 0 \quad\quad\quad\quad 0$$

Because spatial coherence arises from extended sources on propagation, it is intimately connected with the theory of diffraction. The propagation of spatial correlations is described by the <u>Van Cittert- Zernike</u> theorem. We don't have the tools yet to develop this theory any further, but we can gain considerable insight with a simple calculation.

Q: how to measure spatial coherence?
A: just like we did with temporal coherence – with an interferometer.

Of course, if we want to measure spatial correlations, we should use an interferometer which divides a wavefront at different points in space .This is just what Young's double-slit arrangement does.

- Recall plane wave normally incident on a double slit :

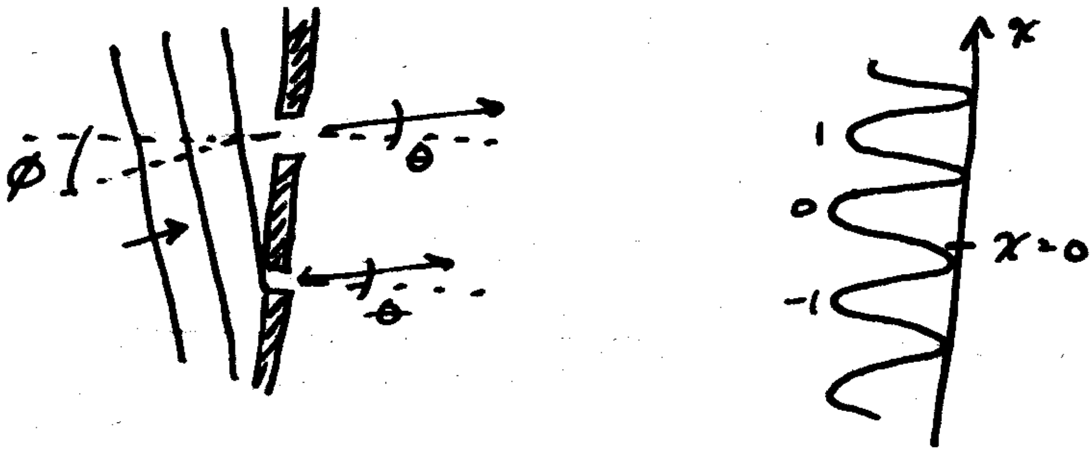


- Zero- order maximum at $\chi = 0$ ($\theta = 0$)
- First-order maximum when OPD= λ

$$h\sin\theta \simeq h\theta = \lambda \Rightarrow \theta = \frac{\lambda}{h}$$

(position on screen: $\tan\theta \simeq \theta = \frac{\chi}{D} \Rightarrow \chi = D\theta = D\frac{\lambda}{h}$.)

Now consider a plane wave incident at an angle φ :



There is now a phase difference $\delta = -\frac{\omega}{c}h\sin\phi$ between the two slits, so the zero-order max. now occurs when

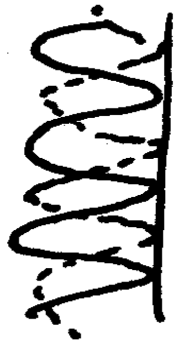
$$-h\sin\phi + h\sin\theta = 0 \quad \text{or} \quad \theta = +\phi$$

1^{st} order : $-h\phi + h\theta = \lambda \Rightarrow \theta = \frac{\lambda}{h} + \phi$

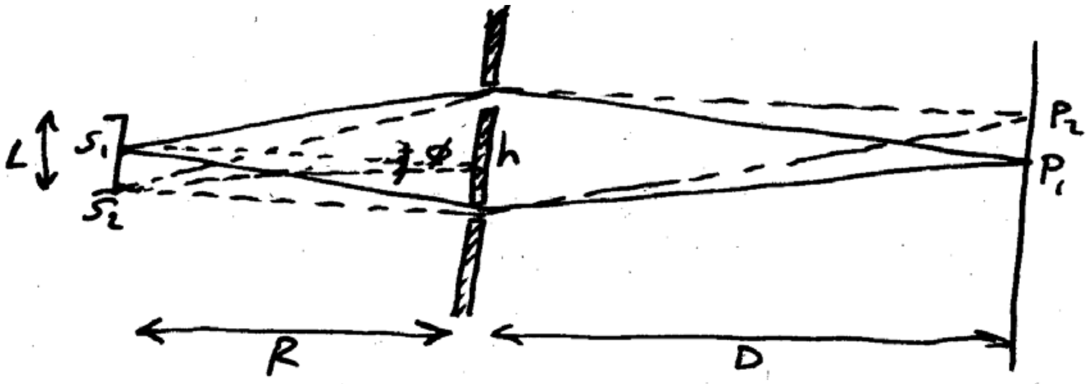
Note that the maxima of the tilted fringe pattern exactly overlap the minima of the $\phi = 0$ (normal) fringe pattern when

$\phi = \pm \frac{\lambda}{2h}$

When this occurs, the interference pattern disappears!



Now we can consider light from two source points on an extended source:



S_1 => fringe pattern centered on P_1

S_2 => fringe pattern centered on P_2

$\phi = \frac{\lambda}{2h} \Rightarrow$ pattern disappears

Geometry: $\tan\phi \simeq \phi = \frac{L/2}{R} = \frac{L}{2R}$

- ⇨ Pattern disappears when the source size is equal or larger than

$$L = 2R\phi = 2R \cdot \frac{\lambda}{2h} = \boxed{\frac{\lambda R}{h} = L}$$

There are other ways of phrasing this which are perhaps more general:

(i) $\phi = \pm \frac{\lambda}{2\mathrm{h}}$ => fringes disappear => no spatial coherence at the two slits

- ⇨ Source is coherent if it subtends an angle smaller than $\Delta\varphi = \frac{\lambda}{\mathrm{h}}$

(ii) The converse of this is perhaps even more useful. If the source subtends an angle $\Delta\phi$, then the transverse coherence length is given by

$$\boxed{l_t = h = \frac{\lambda}{\Delta\phi}}$$ ("coherence area ~ l_t^2)

Since this is the slit separation at which the interference pattern would disappear, where we consider h now to be a variable.

Note as $\Delta\varphi \to 0$, source looks more +more like a point source, and the transverse coherence length becomes very large, as expected.

This is the main result of our discussion.

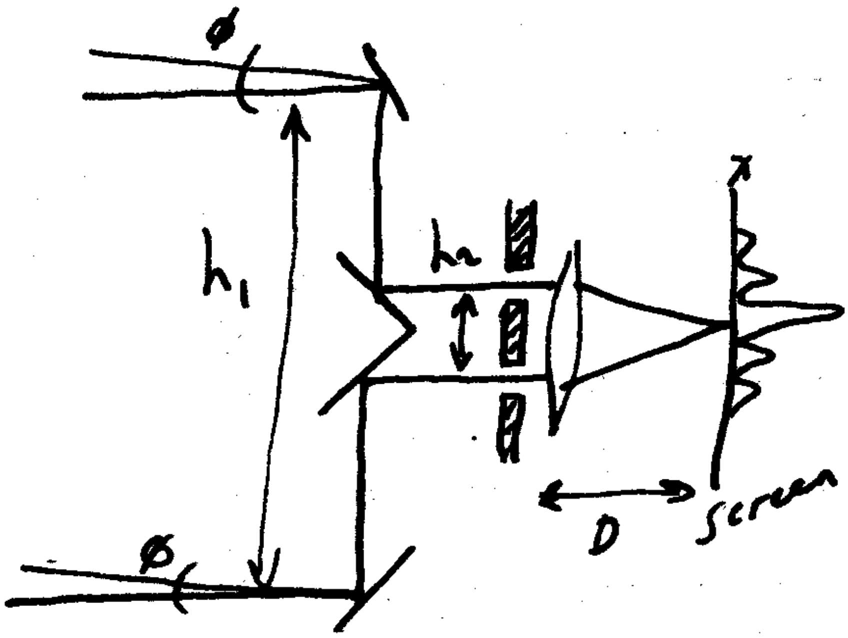
Stellar Interferometer

Michelson realized that the above relation could be used to measure the angular size of stars. The fundamental idea is to vary h, the distance between two sampled points on the wavefronts, and see at what value of h to the fringe disappear.

The naïve approach would be to just use Young's arrangement; the problem is that, for large h, the fringe spacing would get too small ,and the amount of light near the zero order would also be too small.

Michalson's trick: build an interferometer with a fixed slit * separation h_2 (*or pinhole),

But which can sample a wavefront with separation $h = h_1$



- Fringe spacing determined by h_2
- Fringe contrast determined by h_1

- See Guenther for derivation of the intensity on the screen:

$$I_P \approx 2I_0\left[2 - kh_1\Delta\phi\sin(kh_2\frac{x}{D})\right]$$

Ex, star Betelgeuse (red giant in Orion)

$$\Delta\phi \sim 0.047 \quad \text{sec. of arc}$$

corresponds to h_2 on order of 2.5m

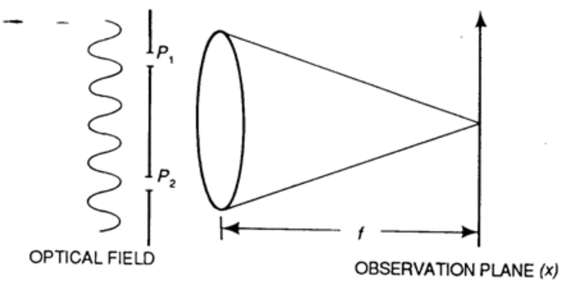


Fig. 10.1 Schematic diagram of the experimental arrangement.

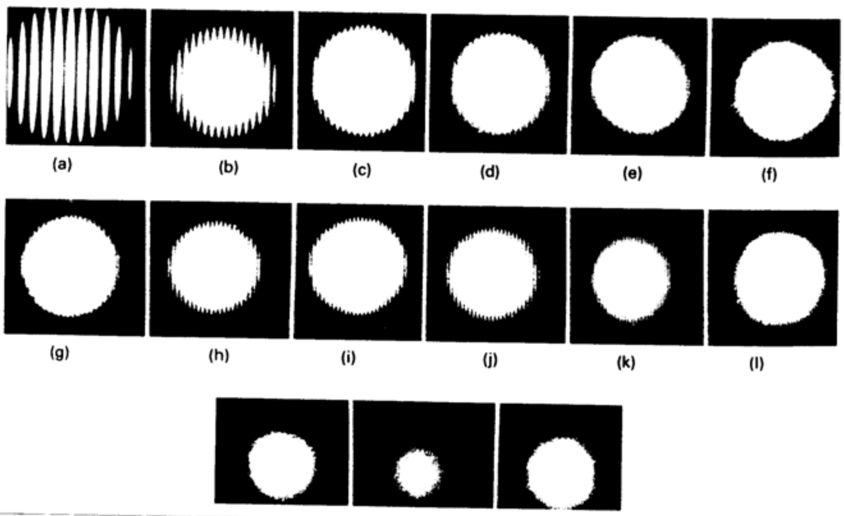
with the diffraction pattern of a circular aperture. We assume that by the use of a microdensitometer we show that it has the form [see Eq. (2.19)]:

$$\left|\frac{2J_1(x)}{x}\right|^2, \tag{10.3}$$

where x is a normalized radial coordinate. We can, therefore, conclude that:

1. The amplitude distribution across the aperture P_1 is uniform.
2. The radiation across the aperture is essentially coherent.

The second aperture, P_2, alone gives a similar result. Now when the two apertures are opened together at their closest separation, two-beam interference fringes are observed that are formed by the division of the incident wavefront by the two apertures. At this closest separation, the fringes are extremely sharp [see Fig. 10.2(a)]. As the separation of the apertures increases, the photographic



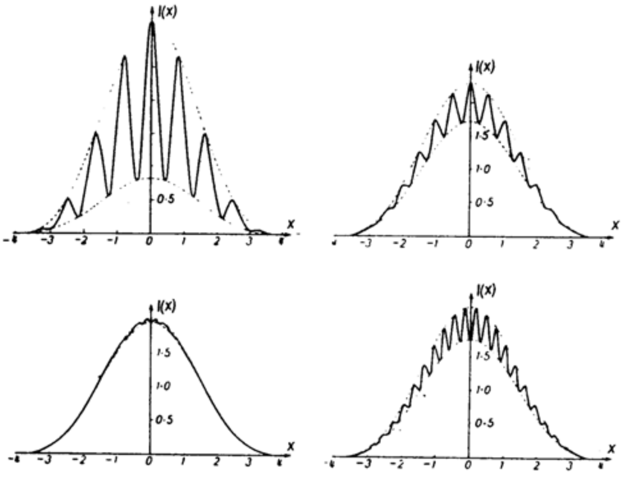


Fig. 10.3 Intensity plots of typical results of Fig. 10.2.

record looks like the results shown in Figs. 10.2(a) through (o). The fringes essentially disappear at (f) only to reappear faintly in (g) through (l), only to fade again at (m), and reappear very faintly at (n) and (o). Intensity plots corresponding to a typical sample of these photographic records are shown in Fig. 10.3. From the results of Fig. 10.2, the following facts are recorded. As the separation of P_1 and P_2 increases,

1. the fringe spacing decreases,
2. the minima are never zero,
3. the relative heights of the maxima above the minima steadily decrease until (f) where they start to increase,
4. the absolute heights of the maxima decrease and the heights of the minima increase until (f),
5. eventually the fringes disappear, at which point the resultant intensity is just twice the intensity observed with one aperture alone, and
6. the fringes reappear with increasing separation but the fringes contain a central minimum not a central maximum.

Items 1 through 5 may be summarized by defining a visibility V [first introduced by Michelson for this very purpose and previously introduced in Chapter 7 as Eq. (7.2)]:

$$V = \frac{I_{max} - I_{min}}{I_{max} + I_{min}} . \qquad (10.4)$$

If this visibility function is plotted against the separation of the apertures P_1 and P_2 for the example given in Fig. 10.2, a curve similar to that shown in Fig. 10.4