Lecture 32

Paraxial Wave Eqn. and Gaussian Beams

Recall from Maxwell's equations we obtained for a time-harmonic wave of freq. ω the Helmholtz eqn. in free space

$$\left(\nabla^2 + k^2\right)E(x,y,z) = 0$$

Where E= (complex) field amplitude of any polarization component of the vector electric field

$$k = \frac{\omega}{c}$$

Let us consider waves that are propagating principally along the z-axis:

$$E(x,y,z) = E_0\psi(x,y,z)e^{-ikz}$$

ψ describes complex transverse profile, and its variation with propagation.

Plug into Helmholtz:

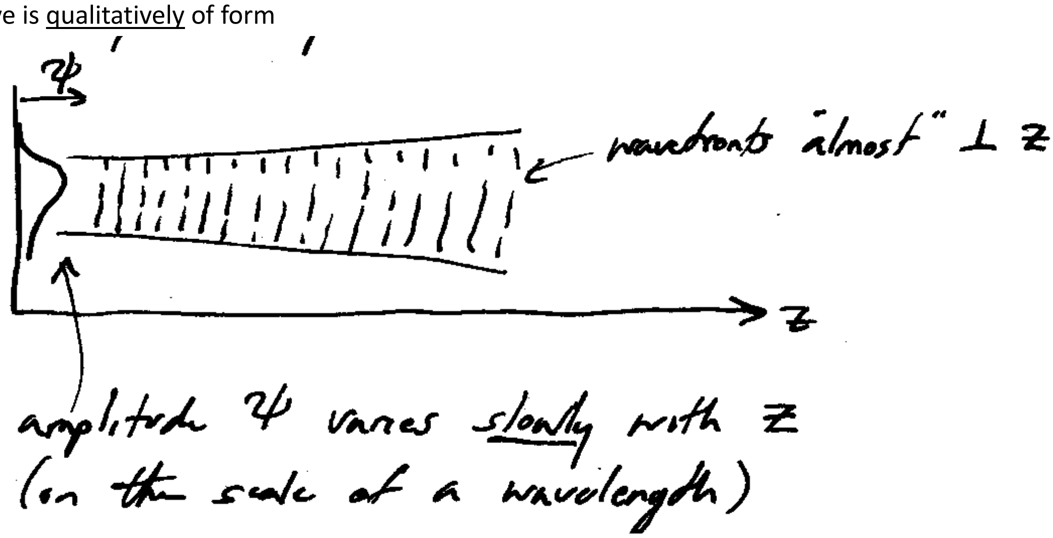
$$\nabla E = \left[\left(\nabla\psi\right)e^{-ikz} - (ik\psi_{\hat{z}})e^{-ikz}\right]E_0$$

$$\nabla^2 E = \left[\left(\nabla^2\psi\right)e^{-ikz} - 2ik\hat{z}(\nabla\psi)e^{-ikz} - k^2\psi e^{-ikz}\right]E_0$$

$$\left(\nabla^2 + k^2\right)E = E_0\left\{\nabla^2\psi - 2ik\frac{\partial\psi}{\partial z}\right\}e^{-ikz} = 0$$

Just as we did for ray optics, we will look for paraxial waves.

⇨ Wave is qualitatively of form



Paraxial approximation:

$$\underbrace{\left|\frac{\partial^2\psi}{\partial z^2}\right| \ll 2k\left|\frac{\partial\psi}{\partial z}\right|}_{\substack{\text{variation of propagation?} \\ \text{slow on scale of}}}, \underbrace{\left|\frac{\partial^2\psi}{\partial z^2}\right| \ll \left|\frac{\partial^2\psi}{\partial x^2}\right|, \left|\frac{\partial^2\psi}{\partial y^2}\right|}_{\substack{\text{variation of propagation?slow} \\ \text{on scale of transverse} \\ \text{extent of wave?}}}$$

⇨ $$\frac{\partial^2\psi}{\partial x^2} + \frac{\partial^2\psi}{\partial y^2} - 2ik\frac{\partial\psi}{\partial z} = 0$$

Or $\boxed{\left(\nabla_T^2 - 2ik\frac{\partial}{\partial z}\right)\varphi(x,y,z) = 0}\ \nabla_T^2 = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}$

=" paraxial wave equation"

Note that this paraxial approximation is equivalent to our earlier, geometrical optics paraxial approximation.

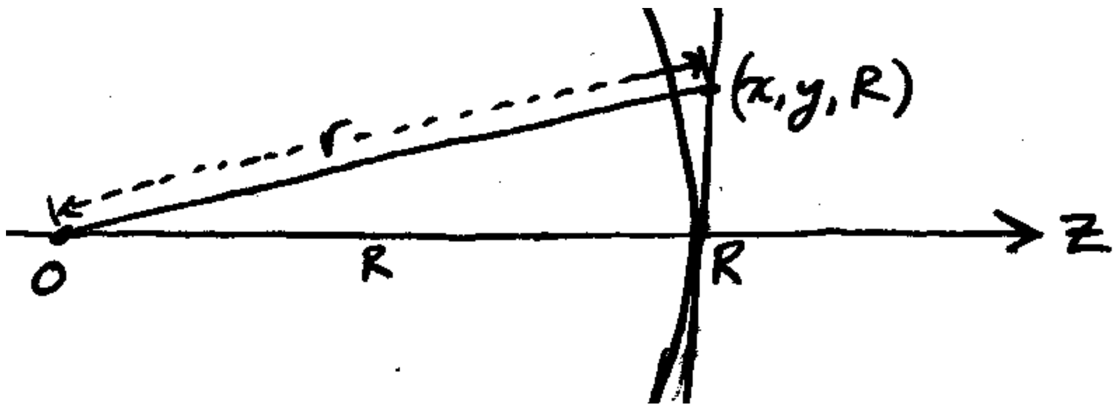
- Rays $=\perp$ to wavefronts => propagation essentially along z => $\sin\theta \simeq \theta$

- see Siegman pp.628-630: he shows there that the paraxial approximation is valid essentially when $\theta < 0.5 rad \leftrightarrow 30°$.

Let's consider a "paraxial wave" more quantitatively now. First consider the spherical wave, which is an exact solution to the Helmholz equation.

$$E(\vec{r}) = A\frac{e^{-ikr}}{r}$$

Now consider the region of this wave propagating along z



$$r = \sqrt{x^2 + y^2 + R^2}$$

In the plane $r = \sqrt{x^2 + y^2 + z^2} = z\left[1 + \frac{x^2 + y^2}{z^2}\right]^{1/2}$, we have

$$E(\vec{r}) = A\frac{e^{-ikr}}{r} = \frac{A}{z\sqrt{1 + \frac{x^2 + y^2}{z^2}}} e^{-ikz\left[1 + \frac{x^2 + y^2}{z^2}\right]^{1/2}}$$

Paraxial approximation again: consider the region near the z axis, so that

$x^2 + y^2 \ll z^2$ $\leftarrow$ Equivalent to $|\tan\theta| \ll 1$

⇨ $$E(\vec{r}) \simeq \frac{A}{z} e^{-ikz\left(1 + \frac{x^2 + y^2}{2z^2}\right)}$$

Where we have neglected $\left(x^2+y^2\right)/z^2$ in the denominator, since it's just a tiny change in the amplitude, but we kept the first-order correction term in the phase in the exponential, since even a small charge in the exponent can change the phase noticeably.

The approximation is often called the Fresnel approximation

At z=R, we have

$$E(\vec{r}) \simeq A\frac{e^{-ikR}}{R}e^{-ik\left(x^2+y^2\right)/2R}$$

This is sometimes called a paraboloidal wave. It's almost a spherical wave, but has a phase that varies quadratically in the transverse direction.

Theorem: The paraboloidal wave obtained by making the Fresnel approximation is an exact analytic solution to the paraxial wave equation.

Proof:

$$E = \psi e^{-ikz}$$

$$\psi\left(x,y,z\right)=\frac{A}{z}e^{-ik\left(x^2+y^2\right)/2z}$$

$$\frac{\partial\psi}{\partial x}=-\frac{ikx}{z}\psi$$

$$\rightarrow \quad \frac{\partial^2\psi}{\partial y^2}=-\frac{ik}{z}\psi-\frac{ikx}{z}\left(-\frac{ikx}{z}\psi\right)=-\frac{ik}{z}\psi-\frac{k^2x^2}{z^2}\psi$$

$$\frac{\partial^2\psi}{\partial x^2}=-\frac{ik}{z}\psi-\frac{k^2y^2}{z^2}\psi$$

$$\frac{\partial^2\psi}{\partial z^2}=-\frac{\psi}{z}+\frac{ik}{2z^2}\left(x^2+y^2\right)\psi$$

Plug in wave eqn.:

$$\nabla_T^2\psi-2ik\frac{\partial\psi}{\partial z}=-\frac{2ik}{z}\psi-\frac{k^2}{z^2}\left(x^2+y^2\right)\psi-2ik\left[-\frac{1}{z}+\frac{ik}{2z^2}\left(x^2+y^2\right)\right]\psi=\underline{\underline{0}} \qquad \underline{\text{exactly}}$$

q.e.d.

Therefore, within the paraxial approximation, near the z axis we have, up to an arbitrary amplitude,

$$E\left(x,y,z\right)=\psi\left(x,y,z\right)e^{-ikz} \quad \text{where} \quad \psi=\frac{1}{z}e^{-ik\left(x^2+y^2\right)/2z}$$

Note that the phase of the paraboloidal wave is

$$\phi\left(x,y,z\right)=\frac{\pi}{\lambda}\frac{x^2+y^2}{z}=\frac{\pi}{\lambda}\frac{x^2+y^2}{R}$$

Which emphasizes that z=R is the radius of curvature of the phase front.

Note also that the choice of origin of our coordinate system must be arbitrary. A wave that satisfies the wave equation in one coordinate system must also be a solution in a displaced coordinate system!

Therefore, the paraboloidal wave solution could just as well be written

$$* \quad \varphi(x,y,z) = \frac{1}{z-\zeta} e^{-ik\left(x^2+y^2\right)/2(x-y)} = \frac{1}{R(z)} e^{-ik\left(x^2+y^2\right)/2R(z)}$$

Where $R(z) = z-\zeta$, and ζ is some arbitrary new origin.

Mathematically, this expression would also be a valid analytic solution to the paraxial wave equation even if ζ were complex – after all, ζ is just a number!

⇨ Write $\zeta = iz_R$, and see what happens!

The complex wave amplitude becomes

$$\psi(x,y,z) = \frac{1}{z+iz_R} e^{-ik\left(x^2+y^2\right)/2(z+iz_R)}$$

Comparing with our earlier expression(*) above , we see that we could define

$$\boxed{q(z) = z + iz_R}$$

As the "complex radius of curvature" of the beam.

Of course, since q is complex, it would be best to separate our the real and imaginary parts in the exponent to get the phase and amplitude separately .Noting that q is in the denominator, we write

$$\frac{1}{q} = \frac{1}{q_r} - i\frac{1}{q_i}$$

⇨
$$\psi = \frac{1}{q} e^{-i\frac{k}{2}\left(x^2+y^2\right)\left(\frac{1}{q_r}-i\frac{1}{q_i}\right)}$$
$$= \frac{1}{q} \underbrace{e^{-ik\left(x^2+y^2\right)/2q_i}}_{\text{looks like a Gaussian}} \underbrace{e^{-ik\left(x^2+y^2\right)/2q_r}}_{\text{looks like our original paraboliodal wave}}$$

The Gaussian amplitude envelope has the advantage over the parabolodal wave that a finite amount of energy is confined near the z axis.

In order to put the Gaussian in a more common form, and to emphasize the similarity of the phase factor with that of the paraboloidal wave, it is useful to write

$$\boxed{\frac{1}{q(z)} = \frac{1}{R(z)} - i\frac{\lambda}{\pi w^2(z)}}$$

So that

$$\boxed{\psi(x,y,z)=\frac{1}{q(z)}e^{-\left(x^2+y^2\right)/w^2(z)}e^{-ik\left(x^2+y^2\right)/2R(z)}}$$

$R(z)$ = （z-dependent） radius of curvature

$w(z)$=spot size (extent of beam transverse to direction of propagation)

The above eqn. describes the "lowest order" (we'll see why) Gaussian beam solution to the paraxial wave eqn.

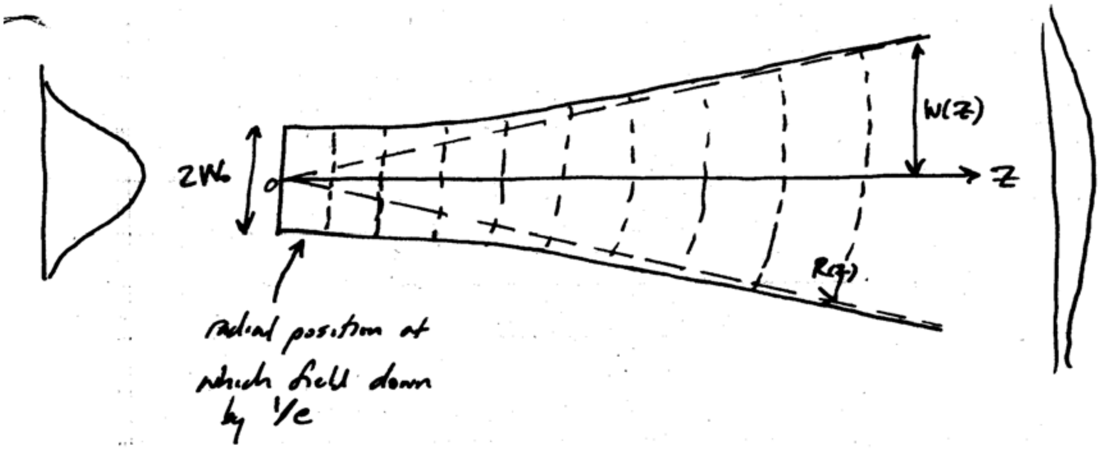
Standard form of Gaussian beam:

- At origin , beam has spot size w_0 and phase front radius $R=\infty$

⇨

$$q_0 = i\frac{\pi w_0^2}{\lambda} \equiv iz_R$$

$$\boxed{z_R = \frac{\pi w_0^2}{\lambda}}$$



After propagation from origin to position z,

$$E(x,y,z)=E_0\sqrt{\frac{2}{\pi}}\frac{q_0}{\omega_0 q(z)}e^{-ikz-ik\left(x^2+y^2\right)/2q(z)}$$

$$=\underbrace{E_0\sqrt{\frac{2}{\pi}}\frac{e^{-ikz+i\phi(z)}}{w(z)}e^{-\left(x^2+y^2\right)/w^2(z)}e^{-ik\left(x^2+y^2\right)/2R(z)}}_{\text{normalized Gaussian}}$$

The beam parameters can be written

$$w(z)=w_0\sqrt{1+\left(\frac{z}{z_R}\right)^2} \qquad \leftarrow \text{ z-dep. Spot size}$$

$$R(z) = z \cdot \left[1 + \left(\frac{z_R}{z}\right)^2\right]$$ ← z-dep. Radius of curvature

$$\phi(z) = \tan^{-1}\left(\frac{z}{z_R}\right)$$ ← "excess" phase delay (Goose effect)

It is important of reiterate that w_0 and z_R are <u>not</u> independent parameters. They are related by $z_R = \frac{\pi w_0^2}{\lambda}$

Thus a Gaussian beam is <u>completely determined</u> by <u>either</u> w_0 or z_R , and λ

w_0 = spot size at focus =>often called the beam waist

<u>Physical meaning</u> of z_R :

- When $z = z_R$,

$$w(z) = w_0\sqrt{1 + \left(\frac{z}{z_R}\right)^2} = \sqrt{2}w_0$$

⇨ Beam has expanded by $\sqrt{2}$ (as measured by the $1/e$ point on the <u>field</u>)

⇨ z_R is a measure of how far the beam is <u>collimated</u> before its divergence due to diffraction becomes significant

- z_R is called the <u>Rayleigh range</u>

- Also common nomenclature: $b = 2z_R$ =<u>confocal parameter</u>