Lecture 24

A complete discussion of image formation using various lens combination may be found in Hecht $\mathbb{S}$5.2.

The procedure is quite straight forward:

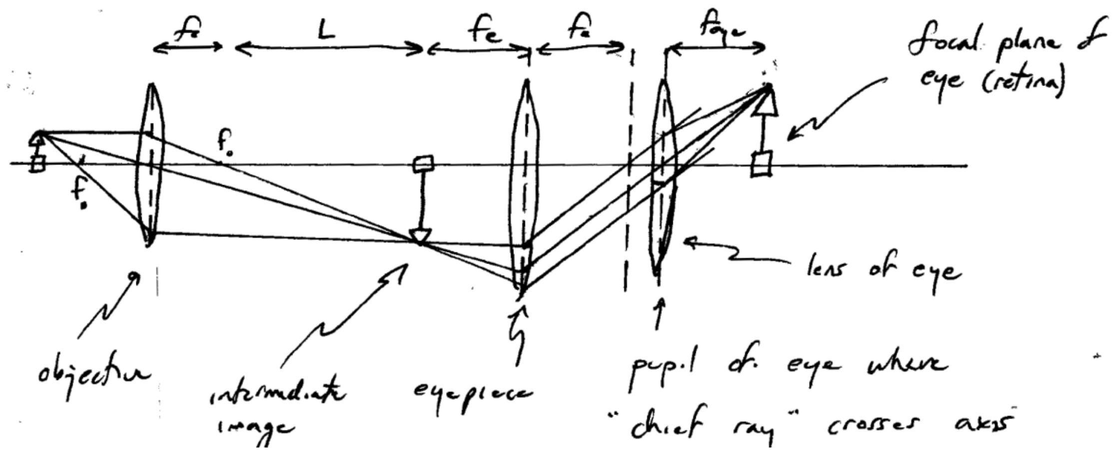
(i) Start with the object and just the first lens

(ii) Find the image formed by the first lens (may be real or virtual)

(iii) Consider the image formed by the first lens to be the object of the second , and find that image

(iv) Repeat sequentially for all the lenses in the system.

Ex. compound microscope(not to scale!)



L- "tube length " = 160mm usually

$\frac{1}{f_{eye}}$ = 60 "diopters" (q diopter = 1 m^{-1})

Ex. two lenses of focal lengths f_1 and f_2 , spaced by a distance d < $f_1 + f_2$

This is considerably trickier –see Hecht fig. $\mathbb{S}$5.29 for an accurate drawing.

The procedure here is to first ignore the second lens and find the image produced by the first lens. Then trace a ray back through the center it is undeviated, and is thus a real ray through both lenses. Then ray 3 on the figure, which is parallel to the axis, goes through the focus of the second lens, and the intersection of rays 2 and 3 gives the location of the final image.

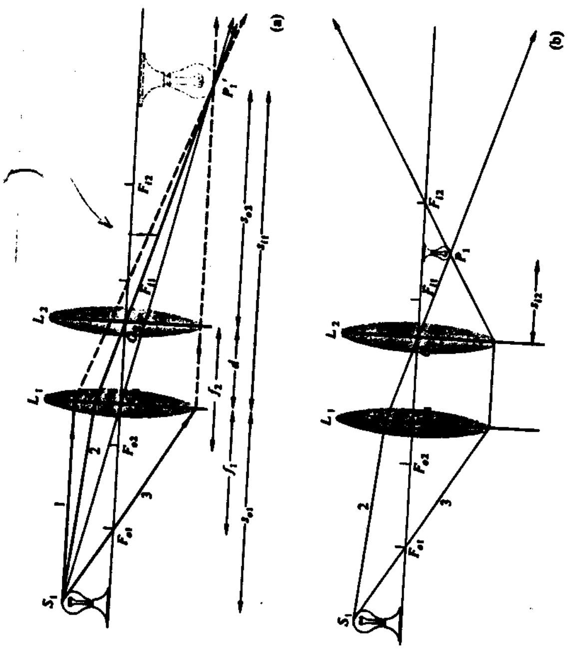


Figure 5.29 Two thin lenses separated by a distance smaller than either focal length.

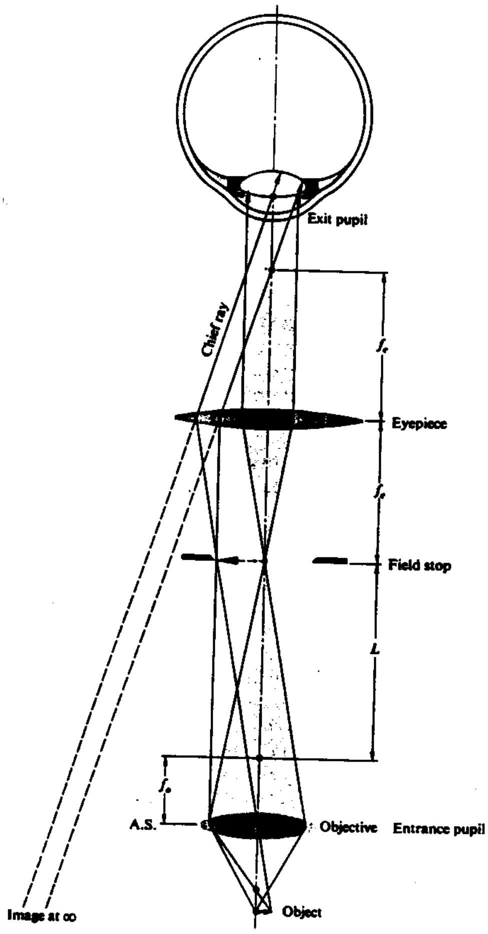


Figure 5.98 A rudimentary compound microscope.

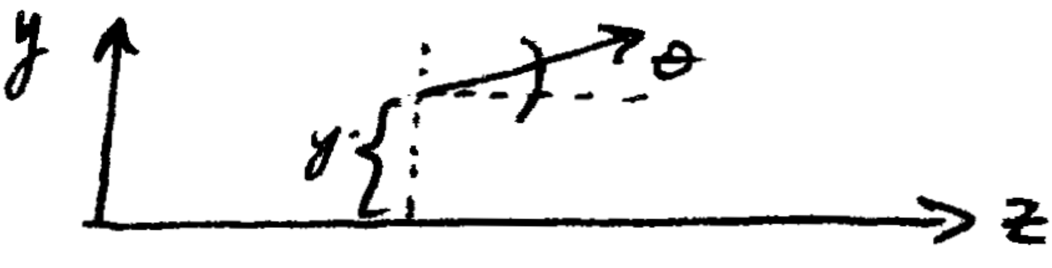
<u>Paraxial Ray Tracing and Matrix Optics</u>

Lipson §3.5

Guenther chap.5 pp.138-144 and 182-190

Siegmen's <u>lasers</u> chap. 15 sections 1 and 2 only (on CTOOLS)

We consider only paraxial rays, propagating at <u>small angles</u> with respect to the z-axis.



θ small => $\sin\theta \simeq \theta$

Clearly, we need to specify only <u>two parameters</u> to completely characterize rays in the <u>meridional plane</u> , namely <u>y and θ</u>.

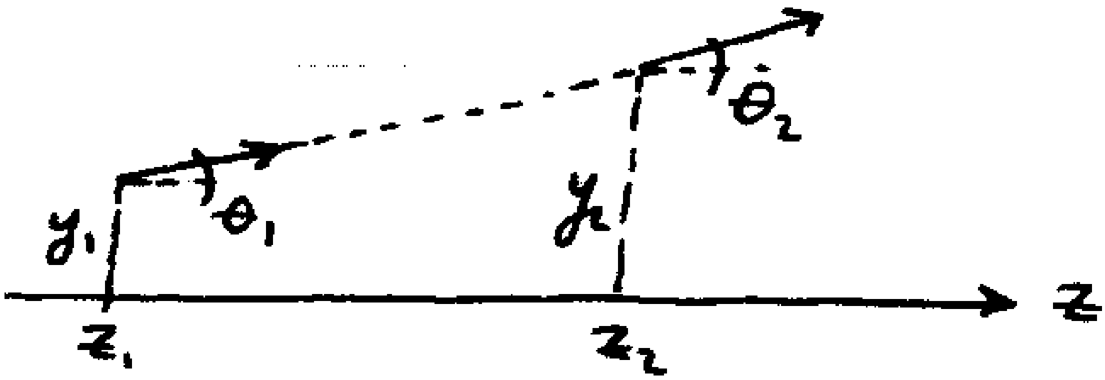
Sign convention: y and θ are positive as shown

(i.e. y>0 = "up" and $\theta > 0$ in ccw direction)

We shall not concern ourselves with the extensions to the theory required to describe skew rays; it will be sufficient for our purposes to consider only meridional rays.

Terminology: we will speak of the ray parameters y and θ at "reference planes" corresponding to z = correct.

1. <u>Propagation in free space</u>



As the ray goes from reference plane z_1 to z_2, θ remains unchanged, and only y changes .

$$\theta_2 = \theta_1$$
$$y_2 = y_1 + (y_2 - y_1) = y_1 + (z_2 - z_1)\tan\theta_1$$

Define d = $z_2 - z_1$ = distance between two reference planes paraxial approximation:

$$\tan\theta_1 \simeq \theta_1$$

⇨ $\boxed{\begin{array}{l} y_2 = y_1 + d\theta_1 \\ \theta_2 = \theta_1 \end{array}}$ " ray transfer equations "

These can be written in the following form:

$$\begin{bmatrix} y_2 \\ \theta_2 \end{bmatrix} = \begin{bmatrix} 1 & d \\ 0 & 1 \end{bmatrix} \begin{bmatrix} y_1 \\ \theta_1 \end{bmatrix}$$

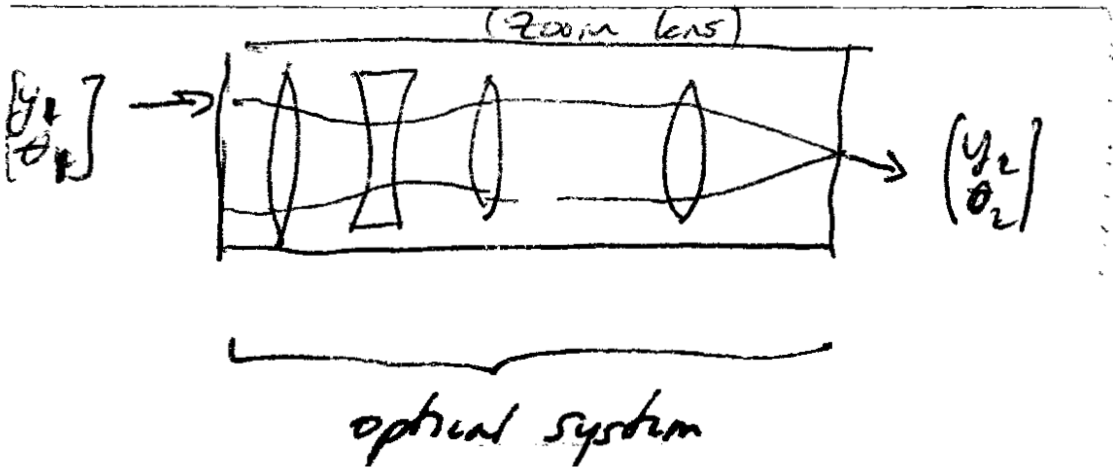
The quantity $\begin{bmatrix} y \\ \theta \end{bmatrix}$ is called a <u>ray vector.</u> (note alternative conviction : $\begin{bmatrix} y \\ n\theta \end{bmatrix}$; this yields slightly different ray matrix from ours)

The matrix $\overleftrightarrow{T} = \begin{bmatrix} 1 & d \\ 0 & 1 \end{bmatrix}$ is the ray transfer matrix of free space.

Our program in ray tracing will follow this same line of treatment. For every optical element, we will determine a ray transfer matrix which will be of the general form

$$\begin{bmatrix} A & B \\ C & D \end{bmatrix}$$

(often called the "ABCD " matrix of the system)
The usefulness of this approach should be obvious:



$$\begin{bmatrix} y_2 \\ \theta_2 \end{bmatrix} = \begin{bmatrix} A_N & B_N \\ C_N & D_N \end{bmatrix} \times \cdots \times \begin{bmatrix} A_3 & B_3 \\ C_3 & D_3 \end{bmatrix} \begin{bmatrix} A_2 & B_2 \\ C_2 & D_2 \end{bmatrix} \begin{bmatrix} A_1 & B_1 \\ C_1 & D_1 \end{bmatrix} \begin{bmatrix} y_1 \\ \theta_1 \end{bmatrix}$$

In other words, we can find the ray transfer matrix of a complicated optical system by simple matrix multiplication , yielding

$$\begin{bmatrix} y_2 \\ \theta_2 \end{bmatrix} = \begin{bmatrix} A & B \\ C & D \end{bmatrix} \begin{bmatrix} y_1 \\ \theta_1 \end{bmatrix}$$

Our first task, then, is to determine the transfer matrices for all the relevant optical elements.

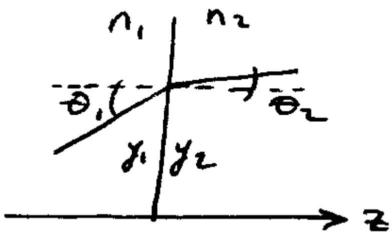
2. Refraction at Planar Interface

$$y_2 = y_1$$

Snell: $n_2 \sin\theta_2 = n_1 \sin\theta_1$

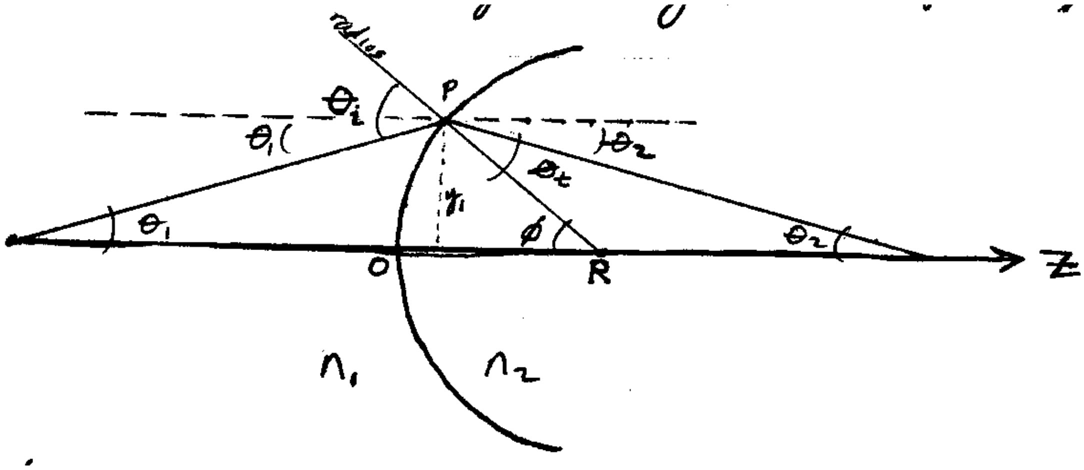
Paraxial: $n_2\theta_2 = n_1\theta_1$

$$\theta_2 = \frac{n_1}{n_2}\theta_1$$



$$\Rightarrow \begin{bmatrix} y_2 \\ \theta_2 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & \frac{n_1}{n_2} \end{bmatrix} \begin{bmatrix} y_1 \\ \theta_1 \end{bmatrix} \Rightarrow \boxed{\begin{bmatrix} A & B \\ C & D \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & \frac{n_1}{n_2} \end{bmatrix}} = \overleftrightarrow{R}$$

3. <u>Refraction at spherical surface</u>
First, we need a <u>sign convention</u>. We follow Guenther +Lipson <u>R>0</u> if the <u>center of curvature</u> is to the <u>right of the vertex</u> , as in the following drawing (Guenther figs 5-9,10):



(vertex of origin O)

Note that the angler θ_1 and θ_2 have been greatly exaggerated for clarity. As drawn, they clearly would not satisfy the paraxial condition!

We use Snell's law at the interface:

$$n_1 \sin\theta_i = n_2 \sin\theta_t \rightarrow n_1\theta_i = n_2\theta_t$$

Now our job is to write θ_i and θ_t in terms of the ray vector angles θ_1 and θ_2 .

$$\theta_i = \theta_1 + \phi$$

$\phi = \theta_t - \theta_2$ (since θ_2 is negative for the ray pointing down)

$$\theta_t = \theta_2 + \phi$$

The ray intersects the sphere at height y_1, so

$\sin\phi = \frac{y_1}{R} \Rightarrow \phi \simeq \frac{y_1}{R}$ (Paraxial approx. again)

=>Snell becomes

$$n_1(\theta_1+\phi)=n_2(\theta_2+\phi)$$

$$n_2\theta_2=n_1\theta_1-(n_2-n_1)\phi=n_1\theta_1-(n_2-n_1)\frac{y_1}{R}$$

$$\theta_2=-\frac{n_2-n_1}{n_2R}y_1+\frac{n_1}{n_2}\theta_1$$

And $y_2=y_1$

The ray transfer matrix for a single spherical surface is therefore just

$$\boxed{\overleftrightarrow{R}=\begin{bmatrix}1 & 0\\ -\frac{n_2-n_1}{n_2R} & \frac{n_1}{n_2}\end{bmatrix}}$$

Digression: we are following a convention in Guenther and many other texts (e.g. Saled+Teich, Photonics), where the ratio of the indices of refraction $\frac{n_1}{n_2}$ appears explicitly in the ray matrix.

Note that for the refraction matrices, we have

$$\det\begin{bmatrix}A & B\\ C & D\end{bmatrix}=AD-BC=\frac{n_1}{n_2}$$

In fact, the determinant of the free space matrix also satisfies this, since $n_1=n_2$!

Theorem: $\det\begin{bmatrix}A & B\\ C & D\end{bmatrix}=\frac{n_1}{n_2}$ for any paraxial system, where n_1 and n_2 are the indices of refraction of the entrance and exit reference planes of the system.

Siegman's Layers and many other texts define the ray matrix in such a way that $\det\begin{bmatrix}A & B\\ C & D\end{bmatrix}=1$ always. In their convention

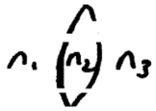
$$\overleftrightarrow{T}=\begin{bmatrix}1 & d/n\\ 0 & 1\end{bmatrix},\overleftrightarrow{R}=\begin{bmatrix}1 & 0\\ -\frac{n_2-n_1}{n_2R} & 1\end{bmatrix}$$; ray vector = $\begin{bmatrix}y\\ n\theta\end{bmatrix}$

It is important in consulting any text or publication to first determinate their convention!

Note: the planar surface result follows from the spherical surface matrix simply by taking $R\to\infty$!

4. Thin lens

   A "thin" lens is one in which we can neglect the thickness (I.e. we don't have to include the propagation in the glass – only the refraction at the surfaces.

   The most general case is easily considered, in which the index of refraction may be different on each side of the lens: 

But for our purposes it will be sufficient to consider a lens in air, so that $n_1 = n_3 = 1, n_2 = n$.

The thin lens matrix is obtained by multiplication

$$\begin{bmatrix} A & B \\ C & D \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ -\frac{1-n}{R_2} & n \end{bmatrix} \begin{bmatrix} 1 & 0 \\ -\frac{n-1}{nR_1} & \frac{1}{n} \end{bmatrix}$$

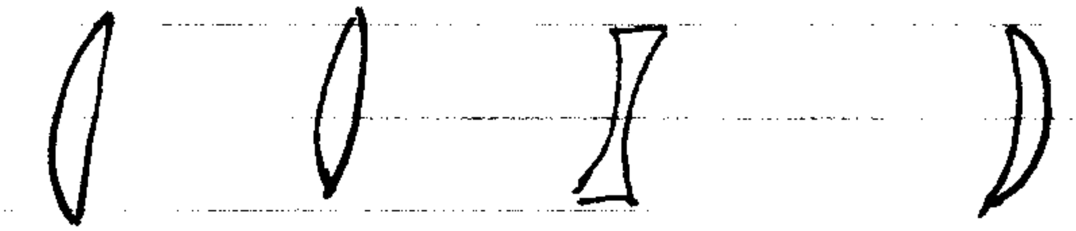
Where R_2 = radius of curvature of second surface

R_1 = radius of curvature of first surface

$$\begin{bmatrix} A & B \\ C & D \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ -\frac{1-n}{R_2} - \frac{n-1}{R_1} & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ -(n-1)\left(\frac{1}{R_1} - \frac{1}{R_2}\right) & 1 \end{bmatrix} = \overleftrightarrow{S}$$

Note that R_1 and R_2 can be positive or negative

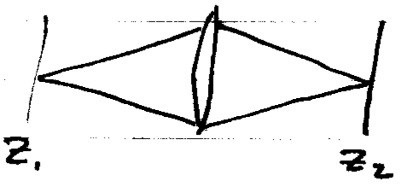
Eg.



Plano-convex	biconvex	biconcave	meniscus
$R_1 > 0$	$R_1 > 0$	$R_1 < 0$	$R_1 < 0$
$R_2 = \infty$	$R_2 < 0$	$R_2 > 0$	$R_2 < 0 \left(R_2 < R_1\right)$

<u>Imaging with one lens</u>

In order to see the significance of the form of $\overleftrightarrow{S}$ above, consider the imaging problem



⇨ We need a matrix taking us from the object to the image plane

$$\overleftrightarrow{O} = \overleftrightarrow{T}_2 \cdot \overleftrightarrow{S} \cdot \overleftrightarrow{T}_1$$

$$\overleftrightarrow{O}=\begin{bmatrix}1 & S'\\0 & 1\end{bmatrix}\begin{bmatrix}1 & 0\\-(n-1)\left(\frac{1}{R_1}-\frac{1}{R_2}\right) & 1\end{bmatrix}\begin{bmatrix}1 & -S\\0 & 1\end{bmatrix}=\begin{bmatrix}1 & S'\\0 & 1\end{bmatrix}\begin{bmatrix}1 & -S\\-(n-1)\left(\frac{1}{R_1}-\frac{1}{R_2}\right) & 1+S(n-1)\left(\frac{1}{R_1}-\frac{1}{R_2}\right)\end{bmatrix}$$

$$=\begin{bmatrix}1-S'(n-1)\left(\frac{1}{R_1}-\frac{1}{R_2}\right) & -S+S'\{1+s(n-1)\left(\frac{1}{R_1}-\frac{1}{R_2}\right)\}\\-(n-1)\left(\frac{1}{R_1}-\frac{1}{R_2}\right) & 1+S(n-1)\left(\frac{1}{R_1}-\frac{1}{R_2}\right)\end{bmatrix}=\begin{bmatrix}A & B\\C & D\end{bmatrix}$$

Recall we have $\begin{bmatrix}y_2\\\theta_2\end{bmatrix}=\begin{bmatrix}A & B\\C & D\end{bmatrix}\begin{bmatrix}y_1\\\theta_1\end{bmatrix}$

For an image to be formed at reference plane z_2, <u>all </u>rays coming from appoint y_1 must end up at a point y_2, <u>independent of θ_1</u>.That means we <u>must have B=0</u> for an <u>image to be formed.</u>

B=0 =>

$$S'+SS'(n-1)\left(\frac{1}{R_1}-\frac{1}{R_2}\right)=S$$

$$(n-1)\left(\frac{1}{R_1}-\frac{1}{R_2}\right)=\frac{1}{S'}-\frac{1}{S}$$

This looks just like the thin lens imaging eqn.

$$\boxed{\frac{1}{S'}-\frac{1}{S}=\frac{1}{\mathrm{f}}}$$

If we identify the focal length as

$$\boxed{\frac{1}{\mathrm{f}}=(n-1)\left(\frac{1}{R_1}-\frac{1}{R_2}\right)}$$

This result is sometimes called the "lens-makers" equation, since it tells you what combinations of refractive index and radii of curvature will produce a lens of a desired focal length.

Back to the ray matrix for a <u>thin lens</u>:

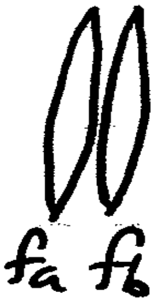
Given the above "lens-makers" equations, we can simplify the form of the matrix $\overleftrightarrow{S}$ on P.205:

$$\boxed{\overleftrightarrow{S}=\begin{bmatrix}1 & 0\\-\frac{1}{\mathrm{f}} & 1\end{bmatrix}}$$

Recall that we assumed n=1 at both the object and image planes. This yields

$$\boxed{\det\overleftrightarrow{S}=1}$$

Example: 2 immediately adjacent thin lenses



The matrix of the combination is

$$\overleftrightarrow{M} = \begin{bmatrix} 1 & 0 \\ -\frac{1}{f_b} & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ -\frac{1}{f_a} & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ -\frac{1}{f_a} - \frac{1}{f_b} & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ -\frac{1}{f} & 1 \end{bmatrix}$$

Where the focal length of the combination is given by

$$\frac{1}{f} = \frac{1}{f_a} + \frac{1}{f_b}$$

Note that without the matrix formalism, the calculation of this would be a mess (see Hecht)! Here it is completely trivial.

Back to the one-lens imaging problem.

We now know that we can rewrite the object-image matrix $\overleftrightarrow{O}$ in the form

$$\overleftrightarrow{O} = \begin{bmatrix} 1 - \frac{S'}{f} & 0 \\ -\frac{1}{f} & 1 + \frac{S}{f} \end{bmatrix}$$

S'= image distance
S= object distance (<0)

Now we want to consider the physical meaning of the off-diagonal elements A and D.