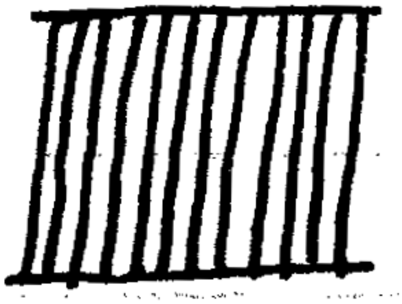
Lecture 37

The Array Theorem

There are many cases in optics, especially the N-slit or diffraction-grating problem, where we must consider diffraction from a number of identical apertures.
e.g. N slits



⇨ Aperture transmission function is (1-D)

$$f(x) = \sum_{n=1}^{N} f_1(x - x_n)$$

where x_n = center position of each aperture

$f_1(x)$ = aperture function of 1 aperture (all identical)

The Fraunhofer diffraction pattern is thus proportional to

$$\varphi(\omega_x) \sim \int_{-\infty}^{\infty} f(x) e^{-iw_x x} dx = \sum_{n=1}^{N} \int_{-\infty}^{\infty} f_1(x - x_n) e^{-iw_x x} dx$$

Now we can write the aperture function as

$$f_1(x - x_n) = \int f_1(x - \alpha)\delta(\alpha - x_n) d\alpha$$

This is the form of a convolution integral. We can then apply the convolution theorem of Fourier transforms.

For any functions $f(x) = \mathbb{F}^{-1}\left[F(w_x)\right]$ and $g(x) = \mathbb{F}^{-1}\left[G(w_x)\right]$

$$\mathbb{F}\left[\int_{-\infty}^{\infty} f(x') g(x - x') dx'\right] = F(w_k) G(w_k) = \mathbb{F}[f(x)] \cdot \mathbb{F}[g(x)]$$

Thus

$$\varphi(w_x) = \mathbb{F}\left[\sum_n f_1(x - x_n)\right] = \sum_n \mathbb{F}[f_1(x - x_n)] = \sum_n \mathbb{F}[f_1(x)] \cdot \mathbb{F}[\delta(x - x_n)]$$

$$\underbrace{\varphi(w_x)}_{\text{total diffraction pattern}} = \underbrace{\mathbb{F}[f_1(x)]}_{\text{diffraction pattern due to one aperture?}} \cdot \underbrace{\mathbb{F}\left[\sum_n^N \delta(x - x_n)\right]}_{\text{interference pattern of N point source with the same spatial distribution as the apertures}}$$

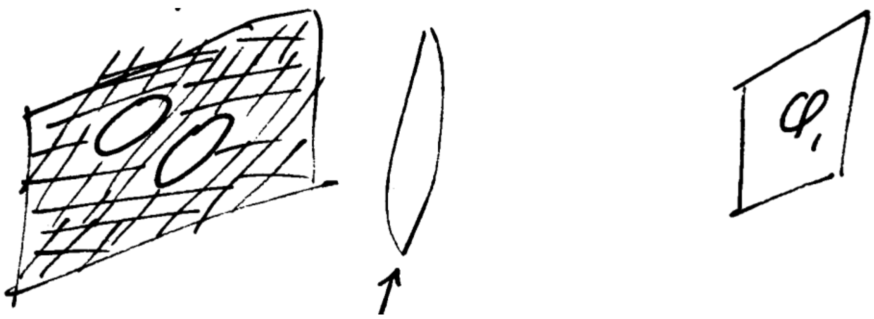
An example of this is the diffraction grating

$$I(\theta_x) = I_0 \cdot \frac{\sin^2 \alpha}{\alpha^2} \cdot \frac{\sin^2 N\beta}{\beta^2} \qquad \alpha = \frac{ka}{2}\sin\theta_x \qquad \beta = \frac{kd}{2}\sin\theta_x$$

Diffraction envelope   interference of N slits

Another useful theorem (we'll show for Fraunhofer diffraction): Babinet's Principle

- Consider an aperture Σ_1

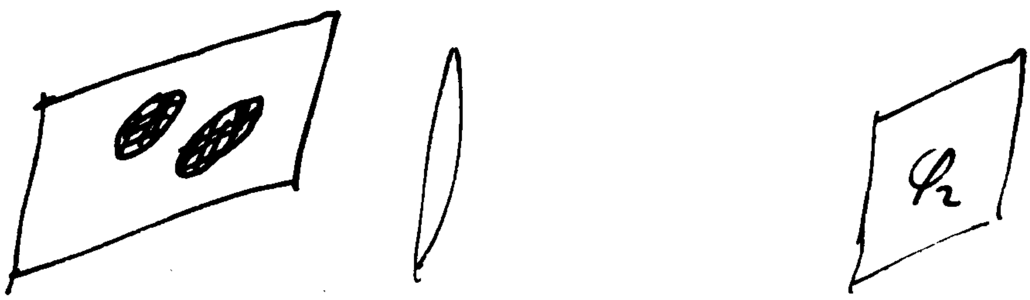


lens to get to far field

φ_1 = diffraction pattern (field!) of Σ_1

Now consider the complementary aperture Σ_2

(the combination of the two apertures is transparent everywhere – i.e. no aperture at all)



Theorem: $\varphi_1 + \varphi_2 = \varphi$, where φ is the field that would be present on the screen with no aperture present.

Proof: left as an exercise (see Guenther app.11-A)

(This is useful on occasion when one knows φ and φ_1, and wants to find φ_2 - e.g. homework corona problem!)

Image formation and resolution

- Lipson 12.1-12.3
- FYI only (not on exam)

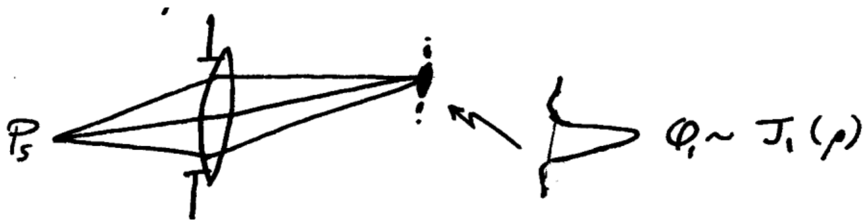
The theory of diffraction we have been developing can be used to give a complete description of image formation. We don't have time to pursue this important theory – the interested student can find the full theory in Goodman's book Fourier Optics (which is used in 435). We shall have to content ourselves with a qualitative consideration of several points of view which arise from the theory.

We know how images are formed in geometrical optics; an ideal imaging system will image each point on an object onto a single point in the image volume. (No aberrations => perfect "stigmatic" imaging.)

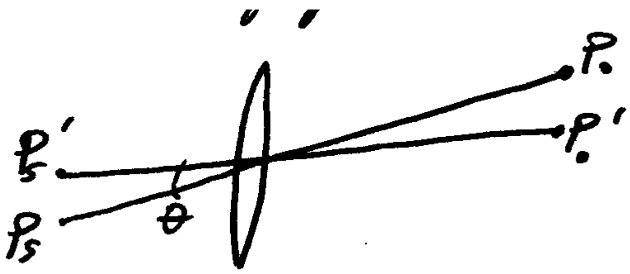
From diffraction theory we know, however, that even in the absence of aberrations, the finite aperture of any real optical system will cause each image point to be blurred (into an Airy disk if the aperture is circular).

Nomenclature:

"point-spread function" = image produced by an optical system of an object point



Now consider the imaging of two object points:



Case(i)： P_s' and P_s are coherent with respect to each other (e.g. object is illuminated with a coherent source such as a laser)

Coherence => add fields

$$\varphi=\varphi_1+\varphi_1'=aJ\left(\rho\right)+\mathrm{b}J\left({}_2\rho\right)$$

Image field = (geometrical image) $\underbrace{\otimes}_{\text{convolution}}$ ($J_1(\rho)$)

Note that the image will depend on the phase relationships of the object points!

Image intensity = $|\text{image field}|^2$

Case (ii): P_s' and P_s are <u>incoherent</u> with each other (e.g. they are independent emitters such as fluorescent molecules ,or they are illuminated by a perfectly spatially incoherent source).

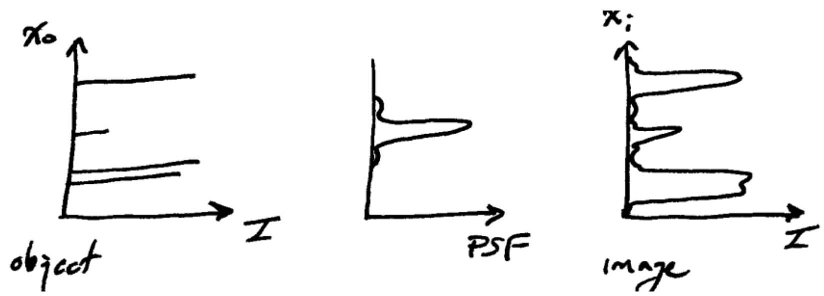
Incoherence => add <u>intensities</u>

$$I = I_1 + I_1' = |a|^2 |J_1|^2 + |b|^2 |J_1'|^2$$

Thus if we consider the point-spread function for <u>incoherent imaging</u> to be the intensity PSF ($=|J_1|^2$ for a circular aperture), we arrive at the statement

image intensity = (geometrical image) $\otimes$ (point spread function)

e.g. I-D imaging



Thus we see that diffraction limits the spatial (or angular) resolution of an imaging system to about the size of the Airy disk.

Resolution:

Rayleigh criterion: two points are said to be resolved if the maximum of the PSF of P_S overlaps with the first minimum of the PSF of $P_S{}'$.

Sparrow criterion: two points are resolved if the intensity shows a minimum between them.

See lipson figs. 12.6+7:

Figure 12.6 Addition of the images of two pinholes incoherently illuminated. The thin lines show the intensity curves and the thick line their sum: (a) Rayleigh and (b) Sparrow separations.

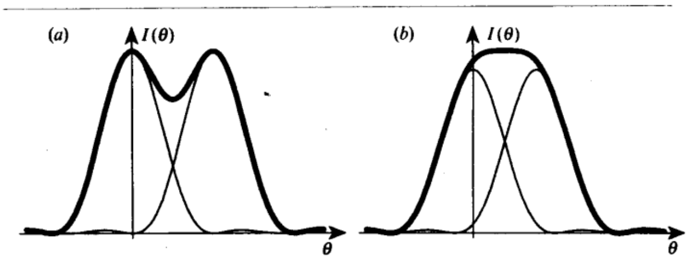
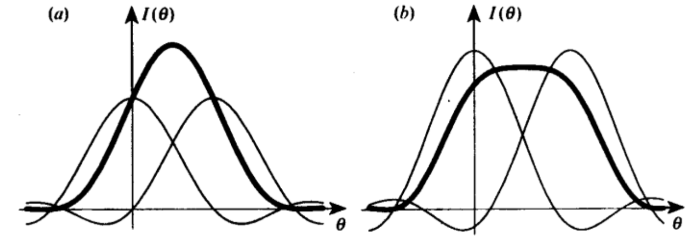


Figure 12.7 Addition of the images of two pinholes coherently illuminated. The thin lines show the amplitude curves and the thick line the square of their sum: (a) Rayleigh and (b) Sparrow separations.



Note the Rayleigh criterion fails for coherent imaging!!

For incoherent imaging, we find from our expression (P.377) for the Airy disk, that the minimum angular separation of two points is

Rayleigh: $\theta_{\min} = 1.22\dfrac{\lambda}{D}$ (D=aperture diameter)

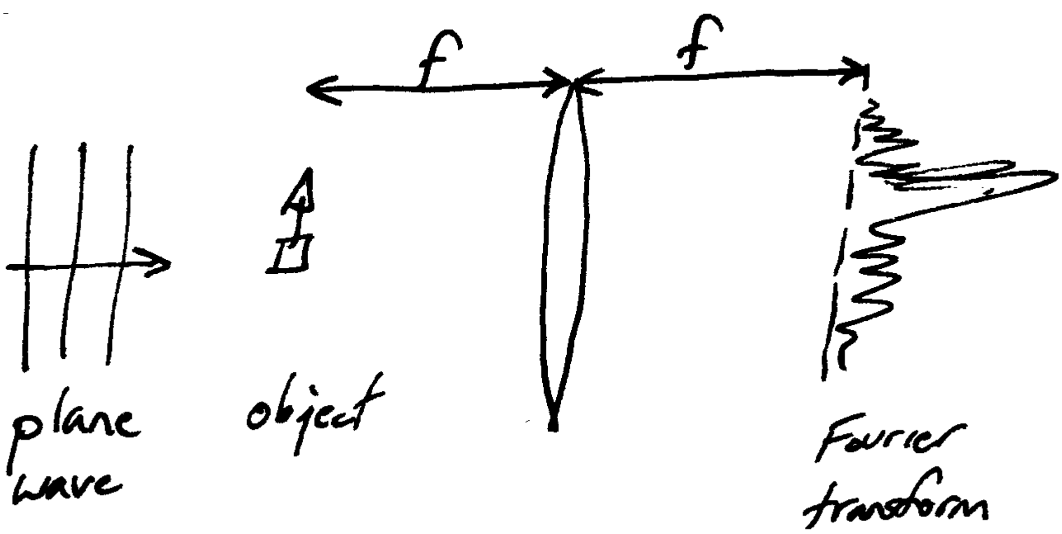
Sparrow: $\theta_{\min} = 0.95\dfrac{\lambda}{D}$

From a Fourier optics point of view, we have seen that a lens acts as a "Fourier transformer" (Fraunhofer diffraction pattern $\propto$ Fourier transform of input field, i.e. aperture function).

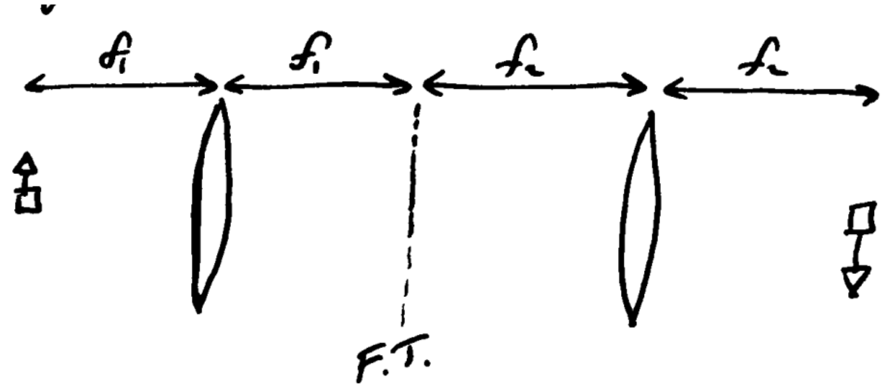
Of course, we saw that the diffraction pattern is not exactly the Fourier transform, but is multiplied by an overall quadratic phase (see P.371).

It is straightforward to show that, if an object is placed at a distance f in front of a lens of focal length f, the quadratic phase is eliminated.

⇨ Field is an exact Fourier transform



From this it is easy to see how a second lens will essentially "inverse Fourier transform" to yield an image (intuition: just apply time-reversed symmetry to propagation!)

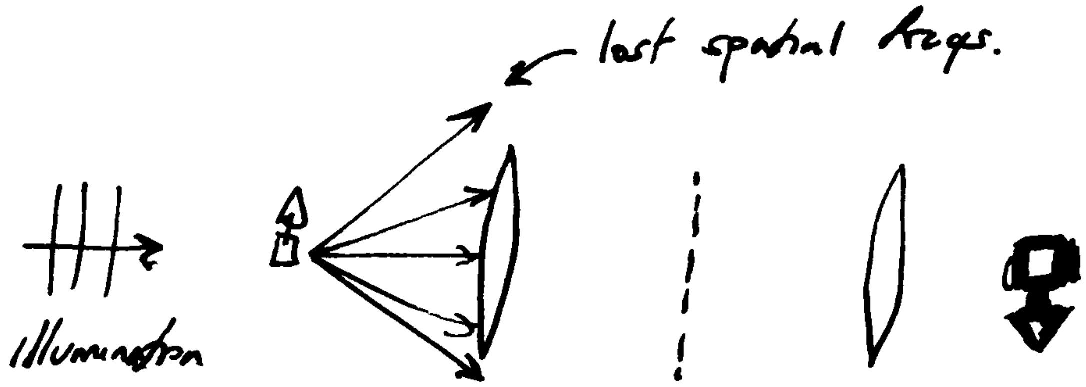


The Fourier analysis shows that the image is magnified and inverted, just as geometrical optics predicts.

⇨ Imaging can be considered to be a process of double Fourier transformation

Effect of finite aperture size: lose high spatial frequencies (see fig. below) => Fourier transform between the two lenses is not exact (missing detail)

⇨ Naturally find image = convolution of object with PSF



si

$$\Delta\phi \sim 0.047\varphi \rightarrow 0 = \frac{\lambda}{\mathrm{h}} \pm l_t^{\,2} = h = h_1 \frac{\lambda}{\Delta\phi} h_2$$

$$I_P \approx 2I_0\left[2 - kh_1\Delta\phi\sin(kh_2\frac{x}{D}\right]$$