Lecture 38

(3) Able's theory of imaging (Lipson 12.2.1)

Consider the object to be a diffraction grating illuminated by a plane wave:

(see Lipson fig.12.1)

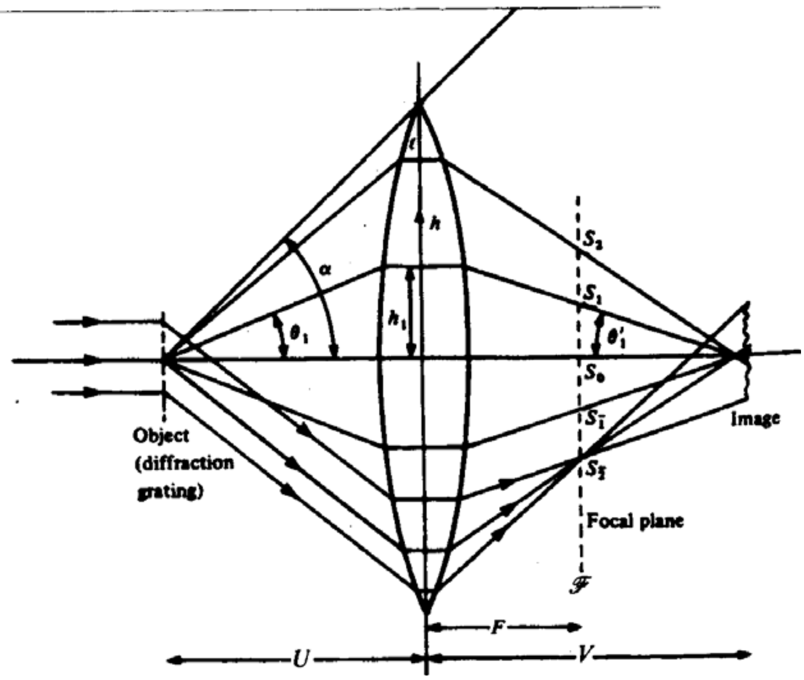
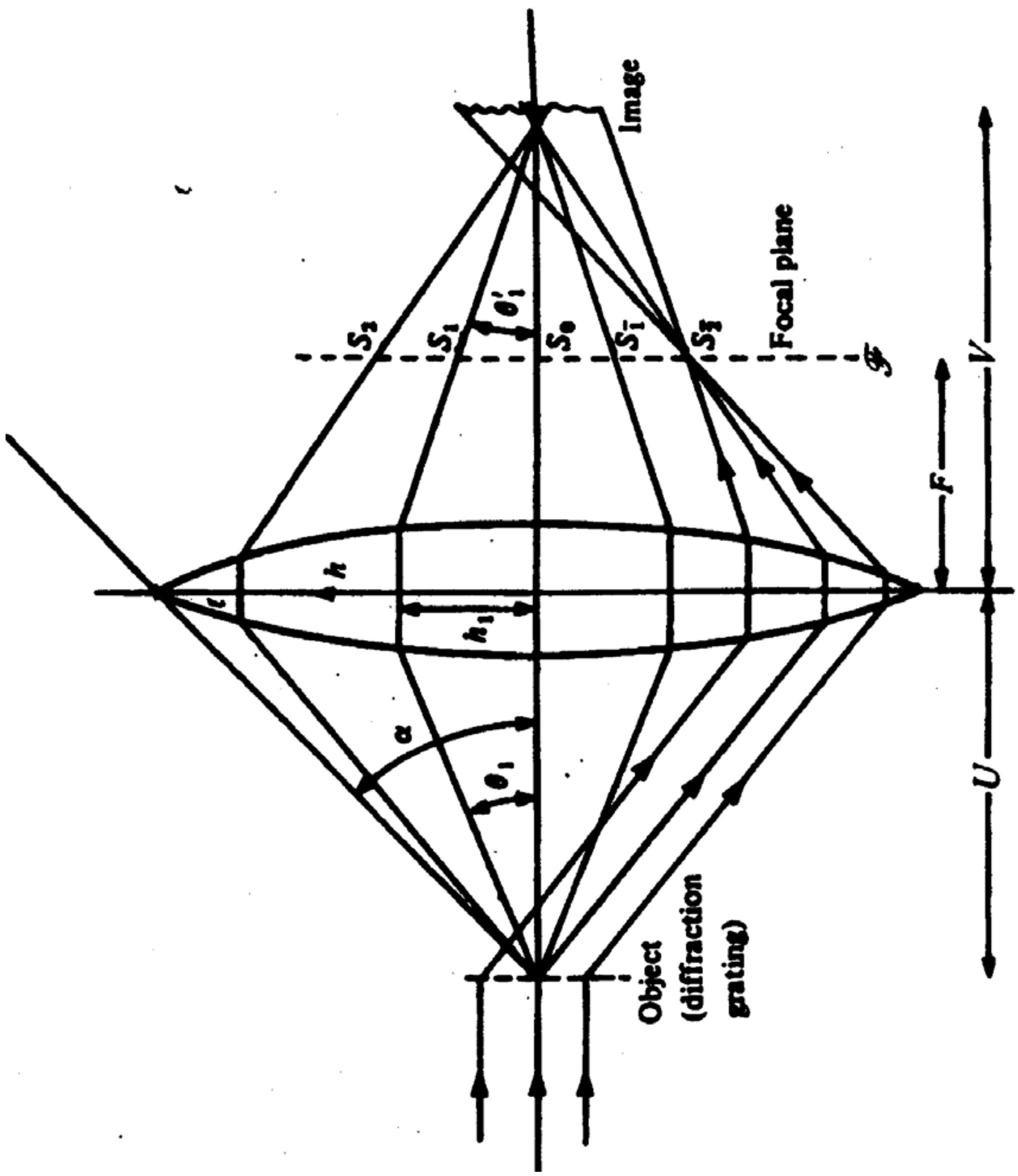


Figure 12.1 Formation of the image of a diffraction grating. Five orders of diffraction are shown, producing five foci S in the plane $\mathscr{F}$. The complete ray bundle is only shown for the -2 order. The angular semi-aperture of the lens is α.

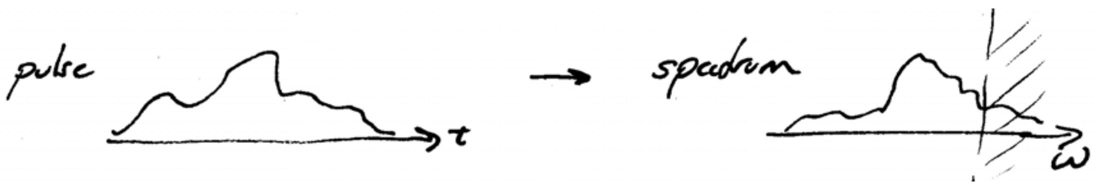
A perfect image of the grating is formed in the plane T if and only if all orders of the light diffracted from the grating are collected by the lens.

Finite aperture =>miss some orders => important image

As we saw on prev. page, this is equivalent to loss of high spatial frequencies.



Consider a time-domain Fourier transform analogy



Truncating the highest frequencies in the spectrum would distort the pulse by removing the fastest feature (low pass filter!)
Spatially , we might decompose an object into spatial frequencies (Fourier transform) , and then see how each spatial frequency is propagated through the system .If spatial frequencies are lost , then fine detail (highest spatial frequency) will be missing from the image => image will be blurred .

(note that in analogy to a pulse, we can conduct any spatial object by an appropriate superposition of gratings with the required spatial frequencies – this is just the Fourier transform relation! – so we can describe imaging entirely in terms of imaging of gratings .This is the Abbe theory of imaging.)
See Lipsons p.12.2.3 for a proof that

$$(\text{image}) = \mathrm{T}^{-1}\left[\mathrm{T}(object)\right]$$

Using the Abbe approach.

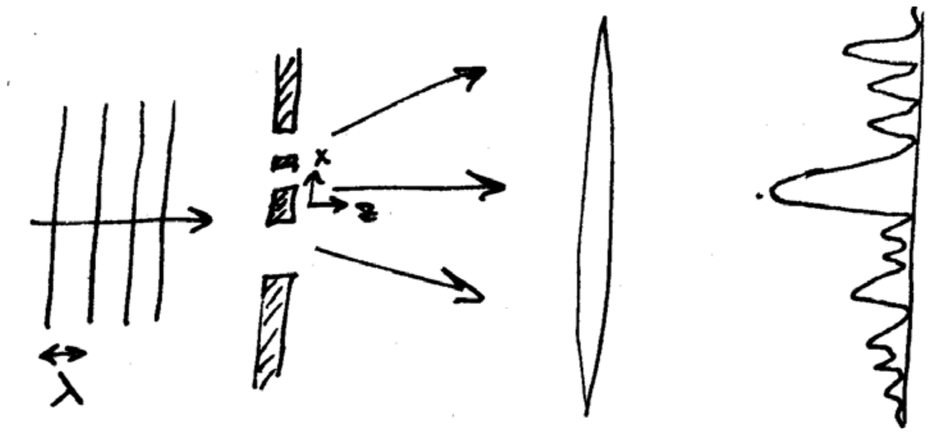
④ Angular spectrum representation (loose)
Let's go all the way back to the Helmholtz eqn.

$$\left(\nabla^2 + k^2\right)\varphi = 0$$

$$\Rightarrow \varphi\left(\vec{r}\right) = e^{-i\vec{k}\cdot\vec{r}} = e^{-i(k_x x + k_y y + k_z z)}$$

Plane wave solution (arbitrary wave $\leftrightarrow$ Fourier sum of these $\leftrightarrow$ "angular spectrum of plane waves ")

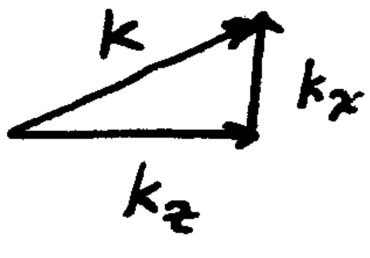
To develop our intuitive picture, we can ignore the y direction and consider only 2-D (x,z) propagation.
The basic idea can be appreciated from a simple problem: consider a monochromatic plane wave incident on an object (e.g. aperture of some scatters in it)



Object scatters light into different angles

Diffraction pattern (far field of lens)

Of course, the scattering does not change the frequency of the light =>



$$k_x^2 + k_z^2 = k^2 = \frac{w^2}{c^2}$$
$$k_z^2 = k^2 - k_x^2$$

Now, suppose the object has structure on a length scale Λ_x.This will give rise to scattered waves in the plane of the aperture with spatial modulation on a scale of Λ_x, with corresponding x-component of the wavevector

$$k_x = \frac{2\pi}{\Lambda_x}$$

Case (i):

$$k_z^2 = k^2 - k_x^2 > 0$$
$$k^2 > k_z^2$$
$$\frac{2\pi}{\lambda} > \frac{2\pi}{\Lambda_x}$$
$$\boxed{\Lambda_x > \lambda}$$

⇨ k_z is <u>real</u> => wave <u>propagates</u> in +z direction as $e^{-ik_z z}$

Case (ii):

$$k_z^2 = k^2 - k_x^2 < 0$$
$$\frac{2\pi}{\lambda} < \frac{2\pi}{\Lambda_x}$$

⇨ $\boxed{\Lambda_x < \lambda}$

⇨ k_z is imaginary

⇨ Defining $\kappa_z = -ik_z$, the wave is evanescent , with the form $e^{-i\kappa_z z}$

Thus we arrive at our main conclusion:

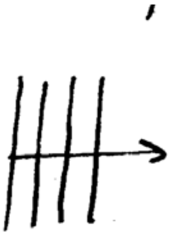
Only wavevectors which propagate to the far field are those for which $k_x < k$ (i.e. $\Lambda_x > \lambda$)

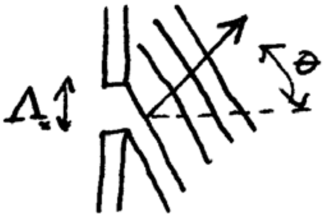
Note that:

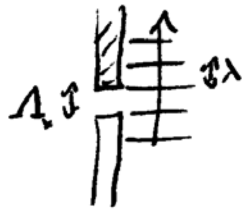
(a) Far-field images can only be formed of object features with spatial frequencies greater than λ (resolution limit)

(b) The case $\Lambda_x = \lambda$ corresponds to a wave running along the x-direction in the aperture

Schematically:







$$k_x = 0$$
$$\rightarrow \Lambda_x = \infty$$

$$k_x = \frac{2\pi}{\Lambda_x}$$
$\Rightarrow$ propagation at angle
$$\sin\theta = \frac{k_x}{k} = \frac{\lambda}{\Lambda_x}$$

$$\Lambda_x = \lambda$$
$$\Rightarrow \theta = 90°$$

(c) remember your Fourier series! You can construct a periodic function with a series of harmonic function, but the finest feature on the function (i.e. the object) is given by the shortest wavelength (highest freq.) in the harmonic series.

Square wave => infinite series

You can think of the waves of different k_x performing a Fourier construction of the object function:

- In plane of aperture, all k_x are possible

⇨ Construct object function to arbitrary detail

- Only these waves with $k_x < k$ propagate

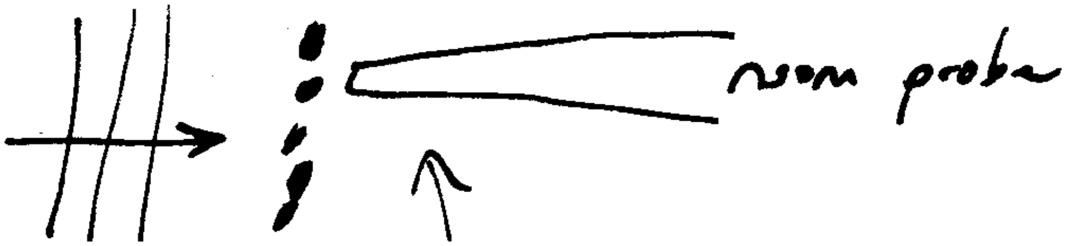
⇨ In Image, you don't have all the Fourier components to reconstruct the image. The high spatial frequencies have been lost => image is blurred.

(d) how can you beat the wavelength –limited resolution in imaging ? Work in the near field, where you have access to the evanescent modes!

This is a hot field of research nowadays, and is called near-field scanning optical microscopy (NSOM).

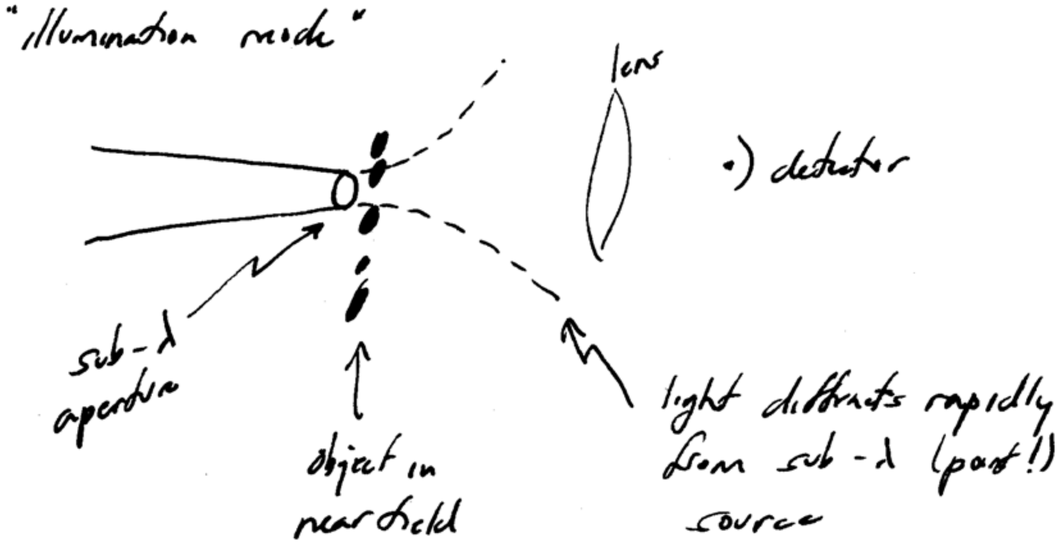
The basic idea is to bring a small probe into the near field of the object which will scatter evanescent waves into propagating waves which can be detected

"collection mode"

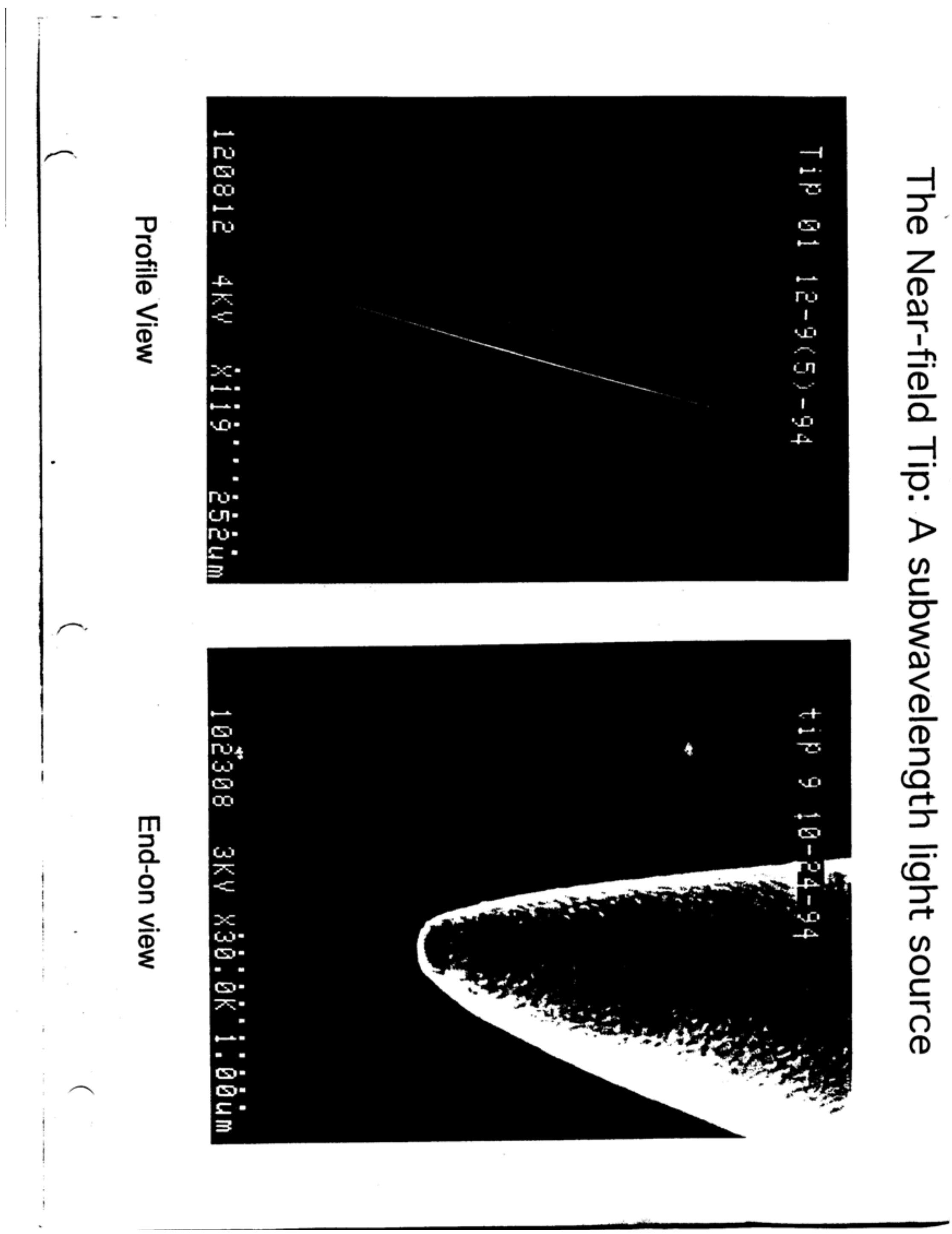


object

probe = fiber with
sub – wavelength aperture
(see next page)



(not to scale!)

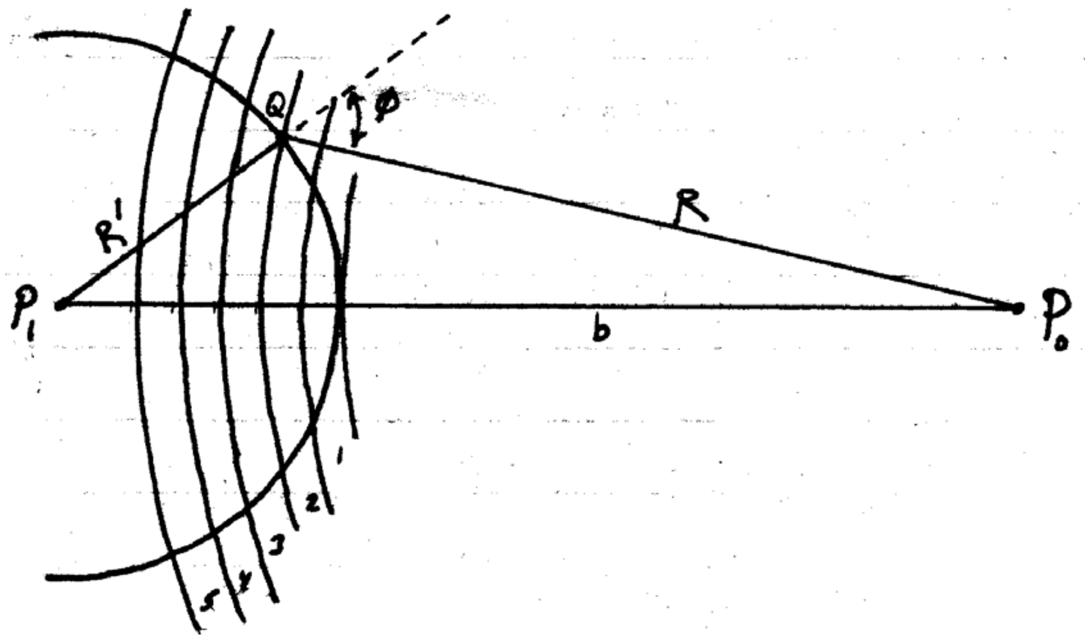


Fresnel Zones

The only remaining concept for us to develop in Fresnel diffraction is that of the Fresnel zone. Guenther has an extensive development of the theory (pp, 444ff), but the idea is very simple and gives considerable insight into the physical origin of diffraction. Our goals here are to understand the basic concept and its major practical consequence – the Fresnel zone plate. It should be noted that one reason zone plates are considered to be important is that they bear a very strong resemblance to a hologram.

The basic idea: <u>Fresnel's construction</u>

- Consider a point source at P emitting spherical waves, observed at point P_0
- Consider a wavefront at radius R'



- The wavefront at radius R' has the form $e^{-ikR'}/R'$.
- Point Q acts as sources of Huygen's wavelets propagating to P_0.
- Along the axis, the wavefront is a distance b from P_0.

⇨ Construct a series of spheres centered on P_0 of radius

$$b, b+\frac{\lambda}{2}, b+\lambda, \ldots, b+\frac{m\lambda}{2}$$

This is known as Fresnel's zone construction.

The index m of the Fresnel zone basically gives the number of half-wavelengths by which the optical path differs from that of the direct line $P_1 - P_0$

- Intuitive idea: light form the 2^{nd} zone is on average π out of phase with respect to light from the first zone, and thus cancels it

⇨ if a circular aperture clips the spherical wave, the intensity at P_0 will be :

-<u>maximum</u> if the number of Fresnel zones is <u>odd</u>
-<u>minimum</u> if the number of Fresnel zones is <u>even</u>

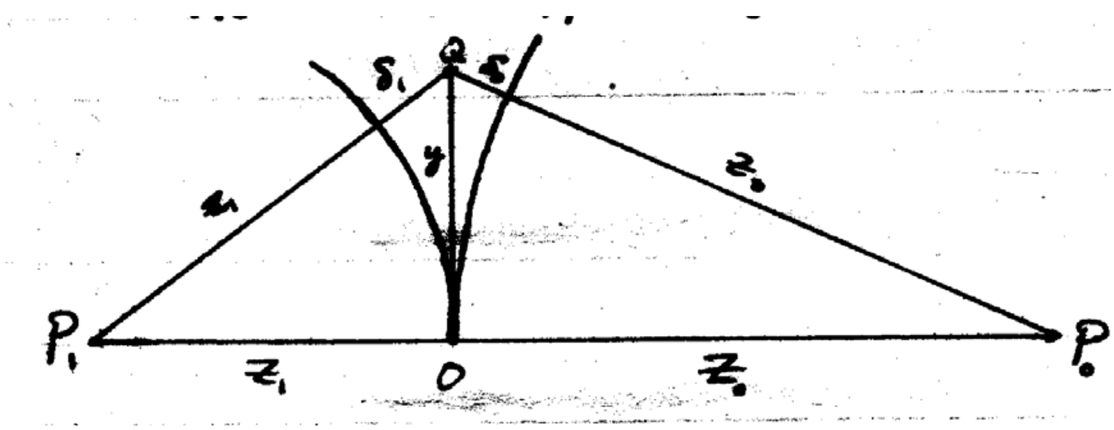
-somewhere in between if a function of a Fresnel zone is exposed by the aperture

How to calculate the total field at P_0 ?

1. construct the zones
2. assume each zone has approximately a constant obliquity factor (can prove is a good assumption)
3. add up the total field due to each zone at P (taking into account the obliquity factor $C(\alpha)$ for each zone)

   The above procedure is followed is Guenther chap.11 (or Born+Wolf see 8.2). Here we will give an approximate calculation of the field at P_0 that is slightly different from the versions given there, but illuminates the essential physics (Young see 5.5.3) :

- consider circular aperture in plane OQ centered on the axis between P_1 and P_0



OPL from P_1 to P_0 on axis = $z_0 + z_1$

OPL $\overline{P_1QP_0} = z_0 + \delta_0 + z_1 + \delta_1$

⇨ OPD= $\delta_0 + \delta_1$

Geometry: $z_1^{\,2} + y^2 = (z_1 + \delta_1)^2 = z_1^{\,2} + 2\delta_1 z_1 + \delta_1^{\,2} \simeq z_1^{\,2} + 2\delta_1 z_1$

Since $\delta_1 \ll z_1$ (small aperture)

$$\Rightarrow y^2 = 2\delta_1 z_1 \Rightarrow \delta_1 \simeq \frac{y^2}{2z_1}$$ "say formula "

- Similarly , $\delta_0 \simeq \dfrac{y^2}{2z_0}$

$$\Rightarrow OPD = \frac{y^2}{2z_0} + \frac{y^2}{2z_1}$$

Define Fresnel Zones by radii y_m where $OPD = \frac{\lambda}{2}, \frac{2\lambda}{2}, \frac{3\lambda}{2}, \ldots$

(*) $$\frac{{y_m}^2}{2}\left(\frac{1}{z_0} + \frac{1}{z_1}\right) = \frac{m\lambda}{2}$$

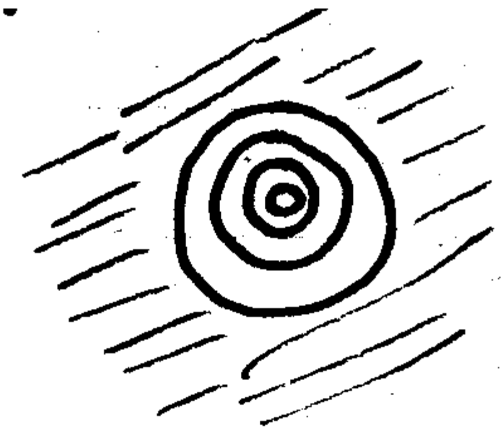
$$y_m = \sqrt{m}\left(\frac{\lambda}{\frac{1}{z_0} + \frac{1}{z_1}}\right)^{1/2} = \sqrt{m}y_1$$

The area of any zone is

$$A_m = \pi {y_m}^2 - \pi {y_{m-1}}^2 = \left[m - (m-1)\right]\pi \frac{z_0 z_1}{z_0 + z_1}\lambda = \pi \frac{z_0 z_1}{z_0 + z_1}\lambda$$ independent of m !

Thus each Fresnel zone has the same area (for a given P_1 and P_0 relative to the aperture plane).

⇨ Each zone contributes an amplitude to the field at P_0 which is just determined by the obliquity factor in that zone C_m



⇨ Total amplitude at P_0 is (for N zones in aperture)

$$\varphi(P_0) \simeq C_1 - C_2 + C_3 - C_4 + \ldots + C_N$$

Lowest approximation:

Now C_m is very closely construct over one zone

$\Rightarrow C_m \simeq C_{m-1}$ also

⇨ $$\left.\begin{aligned}\varphi(P_0) &\simeq C_1, if \text{ N is odd} \\ \varphi(P_0) &\simeq C_1, if \text{ N is even}\end{aligned}\right\}$$ lowest orders approx. (ok form small N)

A better approximation can be derived (Born + Wolf) by rewriting

$$\varphi(P_0) \simeq \frac{C_1}{2} + (\frac{C_1}{2} - C_2 + \frac{C_3}{2}) + (\frac{C_3}{2} - C_4 + \frac{C_5}{2}) + \ldots$$

⇨ Find a more accurate field as

$$\varphi(P_0) \simeq \begin{cases} \frac{1}{2}(C_1 + C_N), \text{N is odd} \\ \frac{1}{2}(C_1 - C_N), \text{N is even} \end{cases}$$

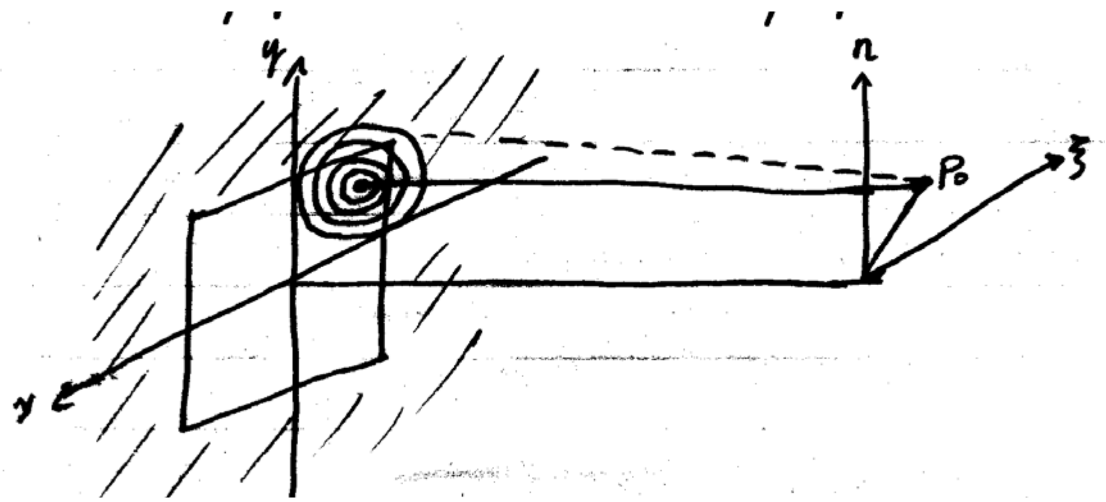
(note C_m decreases with increasing m).

Large N => $C_N \to 0$ (i.e. small)

⇨ Contribution from first zone is twice the amplitude of the unobstructed wave!

Physical origin of diffraction from a Fresnel zone point of view:

- Consider plane wave incident on aperture ($z_1 \to \infty$).
- For any point P_0 in the observation plane , construct Fresnel Zones as shown
- Add contributions due to each zone w/ ± 1 for plane and proportional to area + obliquity factor of each



Note that, the closer the observation point is to the aperture, the more zones there are in the aperture

⇨ The more “fringes” there are in the diffraction pattern, which is what we formed from our Cornel spiral discussion.

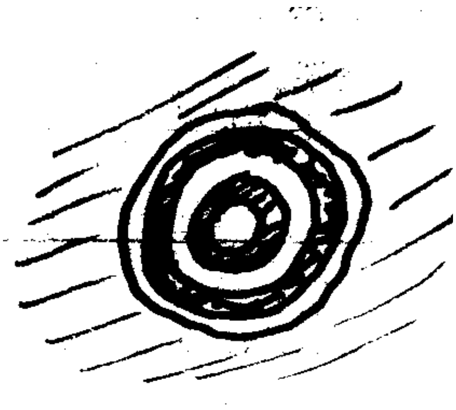
Fresnel Zone plate

We go back to our expression for the field at P_0

$$\varphi(P_0) \simeq C_1 - C_2 + C_3 - C_4 + ... + C_N$$

The even zones tend to cancel (by destructive interference) the fields from the odd zones .Now what happens if we block either the even or the odd zones?
e.g. block the even zones

$$\varphi(P_0) \simeq C_1 + C_3 + C_5 ... + C_N \simeq N\frac{C_1}{2}$$



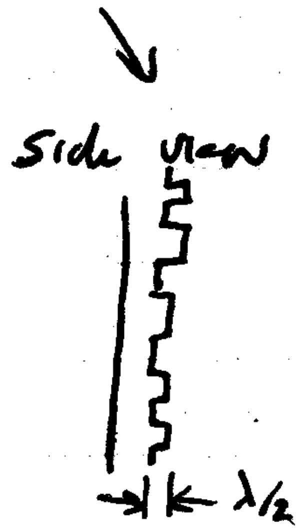
If N is not so large that the obliquity factor varies significantly over the plate

$\Rightarrow \quad I(P_0) \simeq N^2 \frac{I_1}{4} = \left(\frac{N}{2}\right)^2 \times$ contribution of first zone alone

Note: $\frac{N}{2}$ = number of clear zones!

Of course the same thing will happen if the odd zones are blocked; the only difference is the absolute phase of the field at P_0 differs by π.

Such a device, with opaque circles blocking selected zones, is called a zone plate. Note that the intensity at P_0 would be increased still further if, instead of blocking alternate zones, the phases of alternate zones were shifted by π (e.g. by modulating the thickness of a dielectric plate). Exercise for the reader: convince yourself this would increase the intensity at P_0 by a factor of 4.



Now, go back to (*) p.391:

$$\frac{y_m^2}{2}\left(\frac{1}{z_0}+\frac{1}{z_1}\right) = \frac{m\lambda}{2}$$

$$\frac{1}{z_0}+\frac{1}{z_1} = \frac{m\lambda}{y_m^2}$$

We can write $y_m^2 = m y_1^2$ (y_1 = radius of 1^{st} zone)

$$\boxed{\frac{1}{z_0}+\frac{1}{z_1} = \frac{\lambda}{y_1^2}}$$

Now , this is the form of the lens equation if we identify

$z_P = -s$ = object distance

$z_Q = -s'$ = image distance

$f = \frac{y_1^2}{\lambda}$ = focal length => $\frac{1}{s'}+\frac{1}{s} = \frac{1}{f}$ √

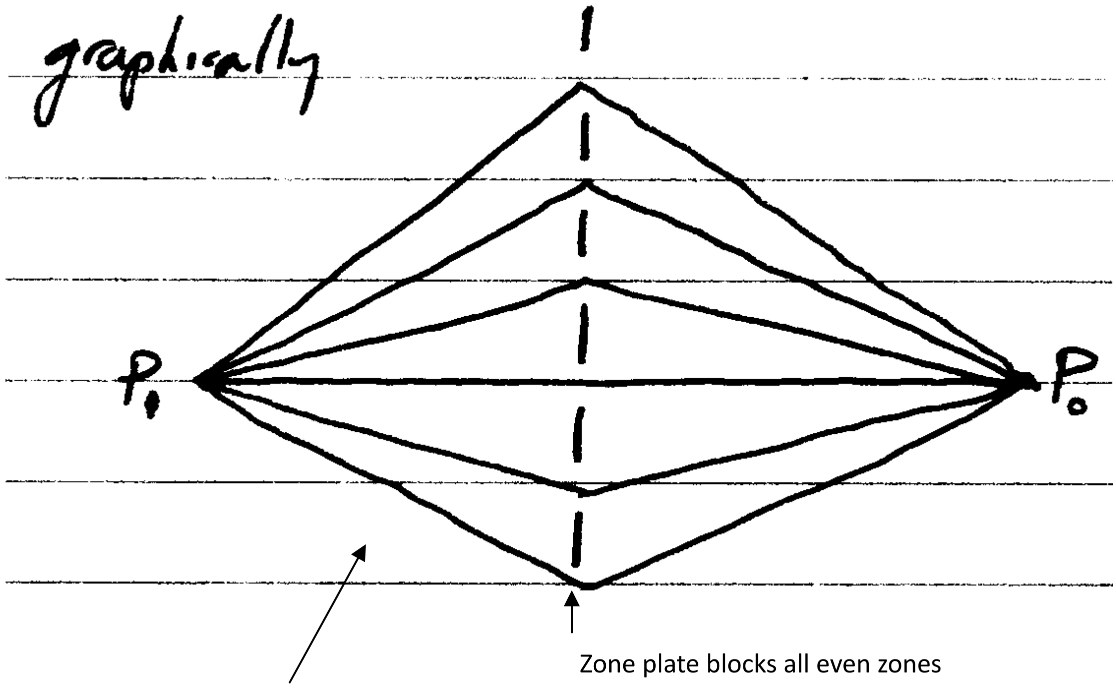
The zone plate thus acts like a lens which images P_1 to P_0!

Note:

1. Focal length depends strongly on λ
   (strong chromatic aberrations!)

2. There isn't only one focus. As you move inward from f towards the aperture , additional Fresnel zones appear, and thus (fainter) images of P_1 will appear at $f/3, \rho/5$,... (proof left to reader)
3. The zone plate has a sense of negative focus in addition to the positive focus (just put P_0 to the left of the aperture)- see Guenther 11-24
4. Zone plate images of extended objects have low contrast

So, given all this, what's it good for ?
1, It can be used to do imaging without lenses. e.g. in x-ray region we don't have any refractive lenses, but plates can be made
2, Analogy to hologram
3, note how we can use the same kind of arguments here about why imaging occurs that we used in our discussion at " Fermat's Principle Revised " on pp.279 ff.(11/9/98)



Zone plate blocks all even zones

All these waves add up in plate at P_0

=>construction interference => higher intensity

Without the zone plate, the only optical path is the stronger line $P_1 \to P_0$, since all others make no contribution due to destructive interference