Lecture 31

The total transmitted field is thus the sum

$$E_{trans} = \left[1 + r_1 r_2 e^{-i\delta} + \left(r_1 r_2 e^{-i\delta}\right)^2 + \dots\right]\left[-t_1 t_2 e^{-i\delta_1}\right] E_{inc} = \left[\sum_{l=0}^{\infty}\left(r_1 r_2 e^{-i\delta}\right)^l\right]\left(-t_1 t_2 e^{-i\delta_1}\right) E_{inc}$$

Now of course $r_1 < 1$ and $r_2 < 1$, so if we define

$$x = r_1 r_2 e^{-i\delta}$$

Then $|x| < 1$, so the infinite series converges

$$1 + x + x^2 + x^3 + \cdots = \sum_{l=0}^{\infty} x^l = \frac{1}{1-x}$$

$$E_{trans} = \frac{-t_1 t_2 e^{-i\delta_i}}{1 - r_1 r_2 e^{-i\delta}} E_{inc}$$

We can get the reflected field in exactly the same way:

$$E_{0r} = r_1 E_{inc}$$

$$E_{1r} = (it_1) r_2 (it_1) e^{-i\delta} E_{inc}$$

$$E_{2r} = r_1 r_2 e^{-i\delta} E_{1r}$$

$$\vdots$$

$$E_{Nr} = \left(r_1 r_2 e^{-i\delta}\right)^{N-1} E_{1r}$$

$$E_{refl} = \left[\sum_{l=0}^{\infty}\left(r_1 r_2 e^{-i\delta}\right)^l\right]\left(-t_1^2 r_2 e^{-i\delta}\right) E_{inc} + r_1 E_{inc}$$

$$E_{refl} = \left[r_1 - \frac{-t_1^2 r_2 e^{-i\delta}}{1 - r_1 r_2 e^{-i\delta}}\right] E_{inc}$$

Now lets' consider the transmitted intensity:

$$I_{trans} = \left|\frac{-t_1 t_2 e^{-i\delta_1}}{1 - r_1 r_2 e^{-i\delta}}\right|^2 I_{inc}$$

This is a general expression, but it will be useful to consider more specifically the special case of the symmetrical Fabry-Perot, where $r_1 = r_2 \equiv r$, and $t_1 = t_2 \equiv t = \sqrt{1-r^2}$.

$$I_{trans} = \frac{t^4}{\left|1 - r^2 e^{-i\delta}\right|^2} I$$

Now

$$\left|1-r^2e^{-i\delta}\right|^2=\left(1-Re^{-i\delta}\right)\left(1-Re^{i\delta}\right)$$
$$=1-R\left(e^{-i\delta}+e^{i\delta}\right)+R^2$$
$$=1-2R\cos\delta+R^2 \qquad\qquad R=r^2$$
$$=1-2R(1-2\sin^2\frac{\delta}{2})+R^2$$
$$=\left(1-R\right)^2+4R\sin^2\frac{\delta}{2}$$

$$I_{trans}=\frac{T^2}{\left(1-R\right)^2\left[1+\frac{4R}{\left(1-R\right)^2}\sin^2\frac{\delta}{2}\right]}I$$

Def. $F=\frac{4R}{\left(1-R\right)^2}$ = "contrast" (sometimes called the coefficient of finesse)

$$\boxed{\frac{I_{trans}}{I_{inc}}=\frac{T^2}{\left(1-R\right)^2}\underbrace{\left[\frac{1}{1+F\sin^2\frac{\delta}{2}}\right]}_{\text{Airy funct}}}$$

Now that when

$$\delta=\frac{2nd\omega}{c}\cos\theta_t=0,2\pi,4\pi,\ldots$$

Then $\sin^2\frac{\delta}{2}=0$

⇨ $I_{trans}=\frac{T^2}{\left(1-R\right)^2}I_{inc}\equiv I_{max}$

If the system is lossless, so T+R=1 , then $I_{max}=I_{inc}$!

⇨ Can write $I_{trans}=\left(\frac{1}{1+F\sin^2\frac{\delta}{2}}\right)I_{max}$

When $\delta=\pi,3\pi,5\pi,\ldots,\sin^2\frac{\delta}{2}=1\Rightarrow$

$$I_{trans}=I_{min}=\frac{I_{max}}{1+F}$$

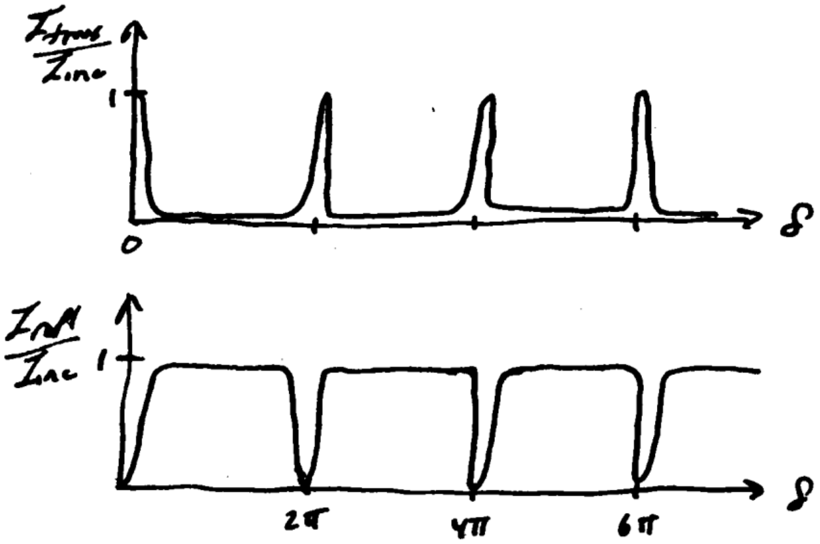
Note that if the mirror reflectance are large, so R is nearly 1, then F becomes very large

=> $I_{min}\ll I_{max}$

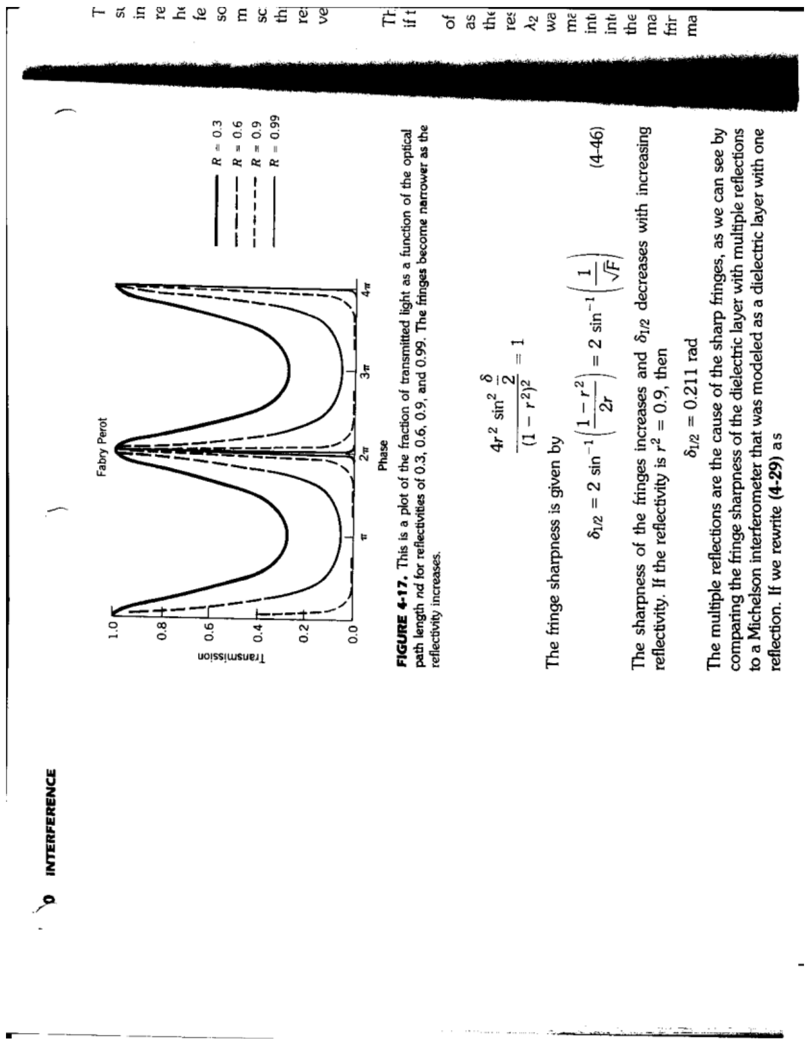
It is easy to verify that for a lossless Fairy-Perot, power is conserved

$$I_{inc} = I_{trans} + I \; ,$$

So we have



See Guenther fig.4-17 for accurate plots with different values of R.



INTERFERENCE

FIGURE 4-17. This is a plot of the fraction of transmitted light as a function of the optical path length *nd* for reflectivities of 0.3, 0.6, 0.9, and 0.99. The fringes become narrower as the reflectivity increases.

$$\frac{4r^2 \sin^2 \frac{\delta}{2}}{(1-r^2)^2} = 1$$

The fringe sharpness is given by

$$\delta_{1/2} = 2\sin^{-1}\left(\frac{1-r^2}{2r}\right) = 2\sin^{-1}\left(\frac{1}{\sqrt{F}}\right) \qquad (4\text{-}46)$$

The sharpness of the fringes increases and $\delta_{1/2}$ decreases with increasing reflectivity. If the reflectivity is $r^2 = 0.9$, then

$$\delta_{1/2} = 0.211 \text{ rad}$$

The multiple reflections are the cause of the sharp fringes, as we can see by comparing the fringe sharpness of the dielectric layer with multiple reflections to a Michelson interferometer that was modeled as a dielectric layer with one reflection. If we rewrite **(4-29)** as

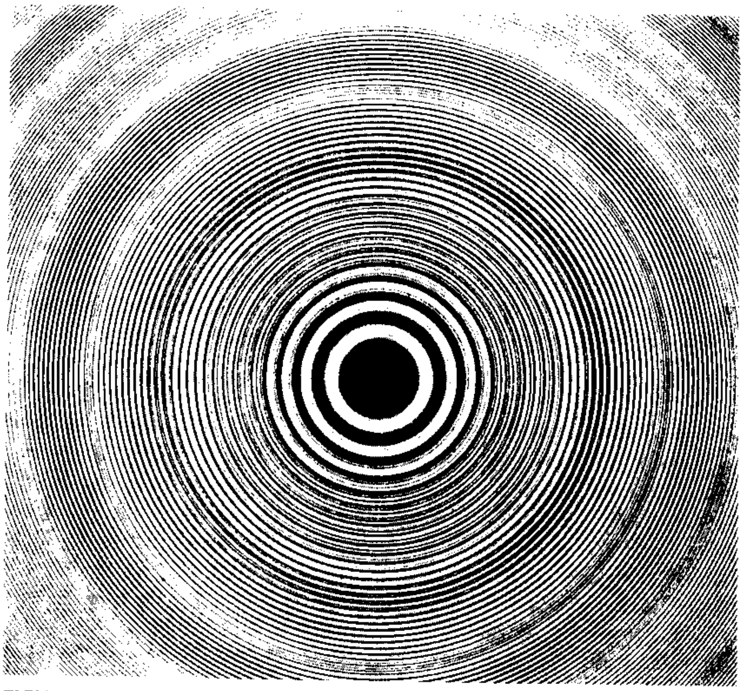


FIGURE 4-19. The output fringes from a Fabry–Perot interferometer. Several sets of frinc due to multiple colors are present in this photo, as can be seen in the print in the color ins Courtesy of Fredrick L. Roesler, University of Wisconsin.

and

$$\sin\left(\frac{\delta_{1/2}}{2}\right) = \frac{\Delta\delta}{4}$$

Using **(4-46)** yields

$$\frac{\Delta\delta}{4} \approx \frac{1}{\sqrt{F}} = \frac{1-r^2}{2r} \qquad (4\text{-}4$$

Differentiating **(4-34)** to get a relationship between $\Delta\delta$ and $\Delta\theta$ yields

$$\Delta\delta = -4\pi n_2 d \sin\theta_t \frac{\Delta\theta_t}{\lambda_0} \qquad (4\text{-}4$$

A bright band will occur whenever

$$2n_2 d\cos\theta_t = m\lambda \qquad (4\text{-}4$$

If we differentiate this

Note that the F-P transmission function is periodic, with a maximum every time

$$\frac{2\omega}{c}\left(nd\cos\theta_t\right) = 2\pi m,\quad m = 0, 1, \ldots$$

The separation in frequency between two orders $(\Delta m = 1)$ is called the free spectral range of the Fabry-Perot :

$$\frac{2\Delta\omega_{FSR}}{c}\left(nd\cos\theta_t\right) = 2\pi$$

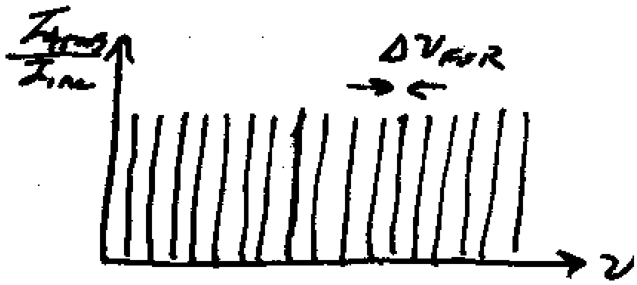
For convenience, we define

$$L = nd\cos\theta_t$$

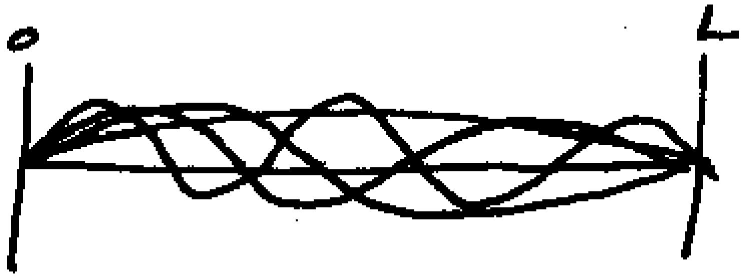
This is obviously just the "effective" thickness" of the Fabry-Perot! (Not the actual O.P.L.!!)

Then $\Delta\omega_{FSR} = 2\pi \frac{c}{2L}$

Or $\boxed{\Delta\nu_{FSR} = \frac{c}{2L}}$



The free spectral range is often called (especially in laser theory) the axial mode spacing of the Fabry-Perot. This is because the transmission resonances simply correspond to the standing waves, or modes, which the cavity can support.



——————————————— (new mode every $\lambda/2$)

Now let's consider the width of the transmission resonances. We will characterize the width as usual by the full-width at half-maximum.
i.e. at what values of δ does

$$\frac{1}{1+F\sin^2\frac{\delta}{2}} = \frac{1}{2}$$

We have $1+F\sin^2\frac{\delta_{1/2}}{2} = 2$

$$F\sin^2\frac{\delta_{1/2}}{2} = 1$$

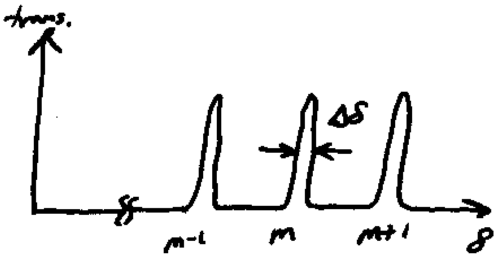
$$\sin\frac{\delta_{1/2}}{2} = \pm\frac{1}{\sqrt{F}}$$

Or $\delta_{1/2} = \pm 2\sin^{-1}\left(\frac{1}{\sqrt{F}}\right)$

Now, when F>>1 (high reflectivity mirrors), as is usually the case , this simplifies considerably , since in this case δ_{v_2} is small

$$\sin\frac{\delta_{1/2}}{2} \simeq \frac{\delta_{1/2}}{2} = \pm\frac{1}{\sqrt{F}} \Rightarrow \delta_{1/2} = \pm\frac{2}{\sqrt{F}}$$

⇨ FWHM $\boxed{\Delta\delta_{1/2} = \frac{4}{\sqrt{F}}}$



It is useful to write this in terms of frequency. Again start with

$$\delta = 2\frac{\omega}{c}\left(n\, d \cos\theta_t\right) = \frac{\omega}{c}2L = 2\pi\nu\cdot\frac{2L}{c}$$

When F is large, $\Delta\delta$ is small, so are may fake the differential of both sides

$$\Delta\delta = 2\pi\Delta\nu\cdot\frac{2L}{c}$$

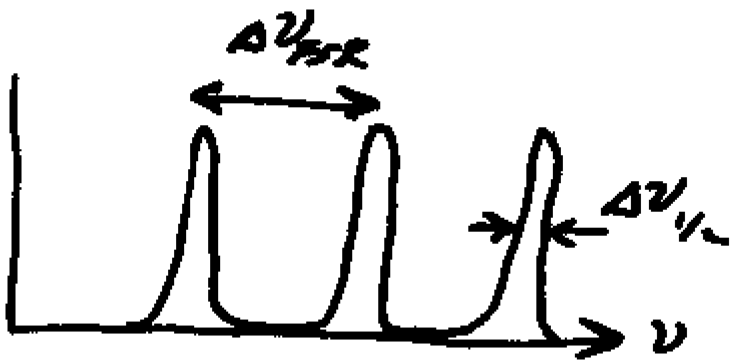
Or $$\Delta\nu_{1/2} = \frac{\Delta\delta_{1/2}}{2\pi}\frac{c}{2L} = \frac{2}{\pi\sqrt{F}}\left(\Delta\nu_{FSR}\right)$$

It is traditional to define the finesse $\mathbb{F}$ by

$$\boxed{\mathbb{F} = \frac{\pi}{2}\sqrt{F}}$$

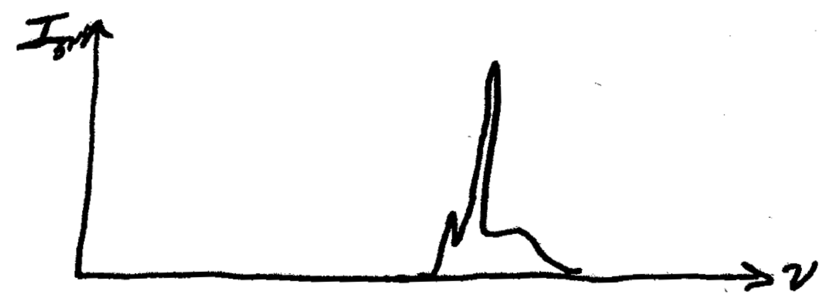
Which yields the ratio of the FWHM to the FSR as

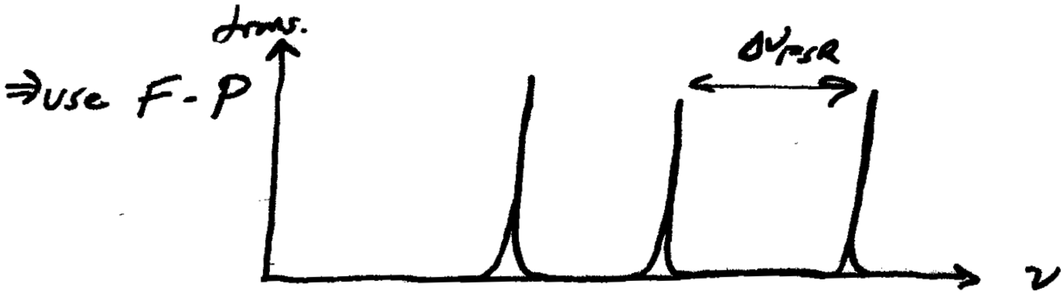
$$\boxed{\frac{\Delta\delta_{1/2}}{\Delta\nu_{FSR}} = \mathbb{F}^{-1}}$$

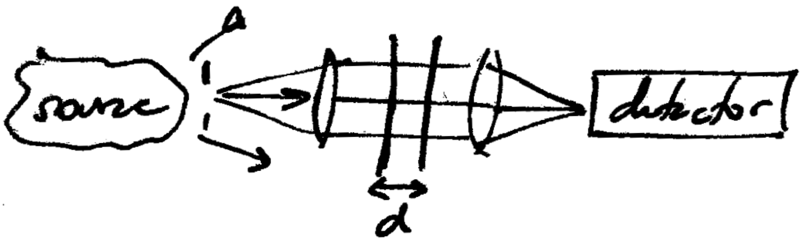


Since $F \gg 1, \mathbb{F} \gg 1$, and $\Delta \nu_{FSR} \gg \Delta \nu_{1/2}$.

Under these circumstances it is possible to use a Fabry-Perot to perform <u>spectroscopy</u>.
e.g. spectrum to be measured







Spectroscopy: measure intensity on detector vs. frequency by scanning the F-P spacing d
Note: we want

(i) $\Delta \nu_{1/2}$ of the F-P to be ,if possible, narrower than the spectral structure of the source to be measured

(ii) $\Delta \nu_{FSR}$ must be larger than the total extent of the source spectrum (the free spectral range causes an ambiguity in the frequency otherwise).

A common implementation is to vary the spacing d electrically via piezoelectric crystals separating the mirrors.

Note convolution: $I_{\text{detector}}(\upsilon) = \int \mathrm{T}_{FP}(\upsilon - \upsilon') I_{\text{source}}(\upsilon') d\upsilon'$

What is the smallest wavelength or frequency interval that can be measured?
The decision of when two wavelengths are resolved is a qualitative, rather than quantitative one, so the choice has some arbitrariness. One common one is to say that two wavelengths are

resolved when the half-amplitude point of one just overlays the half-amplitude point of the other.

Clearly, the separation is then just $\Delta\nu_{1/2}$!

Def. <u>resolving power</u>

$$R = \frac{\lambda}{\Delta\lambda_{1/2}} = \frac{\nu}{\Delta\nu_{1/2}}$$

Note that $\underline{\nu = m\Delta\nu_{FSR}}$ m=order (mode number)

⇨ $$\boxed{R = \frac{m\Delta\nu_{FSR}}{\Delta\nu_{1/2}} = m\mathbb{F}}$$

⇨ To get a high resolving power, use a high $\mathbb{F}$ and high m (but you can't increase m forever without running into trouble with the free spectral range!)