Lecture 25

This looks just like the thin lens imaging eqn.

$$\boxed{\frac{1}{S'}-\frac{1}{S}=\frac{1}{\mathrm{f}}}$$

If we identify the focal length as

$$\boxed{\frac{1}{\mathrm{f}}=(n-1)\left(\frac{1}{R_1}-\frac{1}{R_2}\right)}$$

This result is sometimes called the "lens-makers" equation, since it tells you what combinations of refractive index and radii of curvature will produce a lens of a desired focus length.

Back to the ray matrix for a thin lens:

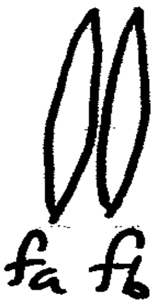
Given the above "lens-makers" equations, we can simplify the form of the matrix $\overleftrightarrow{S}$ on P.205:

$$\boxed{\overleftrightarrow{S}=\begin{bmatrix}1 & 0\\ -\frac{1}{\mathrm{f}} & 1\end{bmatrix}}$$

Recall that we assumed n=1 at both the object and image planes. This yield:

$$\boxed{\det\overleftrightarrow{S}=1}$$

Example: 2 immediately adjacent thin lenses



The matrix of the combination is:

$$\overleftrightarrow{M}=\begin{bmatrix}1 & 0\\ -\frac{1}{\mathrm{f}_b} & 1\end{bmatrix}\begin{bmatrix}1 & 0\\ -\frac{1}{\mathrm{f}_a} & 1\end{bmatrix}=\begin{bmatrix}1 & 0\\ -\frac{1}{\mathrm{f}_a}-\frac{1}{\mathrm{f}_b} & 1\end{bmatrix}=\begin{bmatrix}1 & 0\\ -\frac{1}{\mathrm{f}} & 1\end{bmatrix}$$

Where the focal length of the combination is given by

$$\frac{1}{\mathrm{f}}=\frac{1}{\mathrm{f}_a}+\frac{1}{\mathrm{f}_b}$$

Note that without the matrix formalism, the calculation of this would be a mess (see Hecht)! Here it is completely trivial.

Back to the one-lens imaging problem.

We now know that we can rewrite the object-image matrix $\overleftrightarrow{O}$ in the form

$$\overleftrightarrow{O}=\begin{bmatrix} 1-\dfrac{S'}{f} & 0 \\ -\dfrac{1}{f} & 1+\dfrac{S}{f} \end{bmatrix}$$

S'= image distance

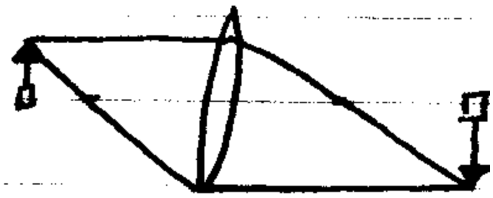
S= object distance (<0)

Now we want to consider the physical meaning of the off-diagonal elements A and D.

$$\begin{bmatrix} y_2 \\ \theta_2 \end{bmatrix}=\begin{bmatrix} 1-\dfrac{S'}{f} & 0 \\ -\dfrac{1}{f} & 1+\dfrac{S}{f} \end{bmatrix}\begin{bmatrix} y_1 \\ \theta_1 \end{bmatrix}$$

⇨ $y_2=\left(1-\dfrac{S'}{f}\right)y_1$

All rays that start at height y_1 end up at y_2.



Clearly, $\boxed{\beta\equiv\dfrac{y_2}{y_1}}$ is the lateral magnification of the system.

Note that

$$\beta=1-\frac{S'}{f}=1-S'\left(\frac{1}{S'}-\frac{1}{S}\right)=1-1+\frac{S'}{S}$$

$$\boxed{\beta=\frac{S'}{S}}$$

i.e. the familiar result that the magnification of a lens is just the ratio of the image and object distances.

Note that S<0, so that if the image is real, corresponding to $s'>0$, then $\beta<0$

⇨ The image is inverted.

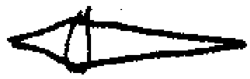
The only remaining parameter in $\overleftrightarrow{O}$ is D: note that

$$1+\frac{S}{f}=1+S\left(\frac{1}{S'}-\frac{1}{S}\right)=1+\frac{S}{S'}-1=\frac{S}{S'}=\frac{1}{\beta}$$

Thus, for an object point on-axis ($y_1 = 0$), we have

$$\theta_2 = \frac{1}{\beta}\theta_1$$

We see that, while the lateral dimensions of the object are magnified by β, there is an angular



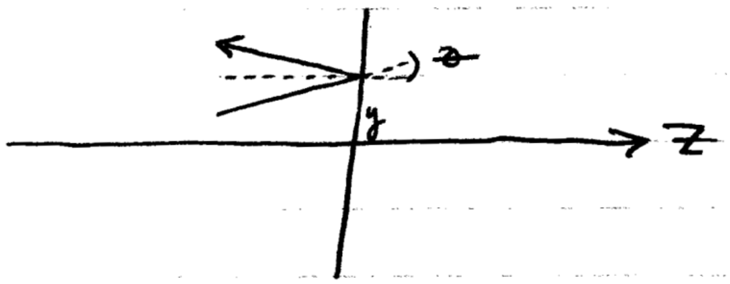
demagnification $\frac{1}{\beta}$ of the rays that occurs simultaneously.

The final form of the object–image matrix is therefore:

$$\boxed{\overleftrightarrow{O} = \begin{bmatrix} \beta & 0 \\ -\frac{1}{\mathrm{f}} & \frac{1}{\beta} \end{bmatrix}}$$

Some other useful matrices: (proof left to readers)

- Reflection form planar mirror



Sign convention: z>0 is defined by the direction of propagation

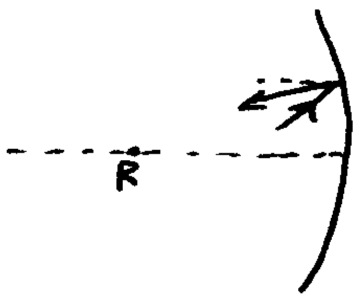
⇨ $\overleftrightarrow{M} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$

- Reflection from spherical mirror of radius R convention

R<0 ↔ concave

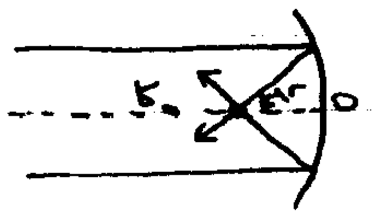
R>0 ↔ convex

$$\overleftrightarrow{M} = \begin{bmatrix} 1 & 0 \\ \frac{2}{R} & 1 \end{bmatrix}$$



Note this looks just like the thin lens matrix, but with $f = -\frac{R}{2}$

(i.e. a concave mirror is a positive lens)



<u>quadratic gradient-index medium</u> (Siegman's term ="duct")

- see previous lecture (note P.185); there we derived that

$$\frac{d^2y}{dz^2}+\alpha^2 y=0 \quad \text{(in paraxial approx.)}$$

⇨ for $\alpha^2>0$

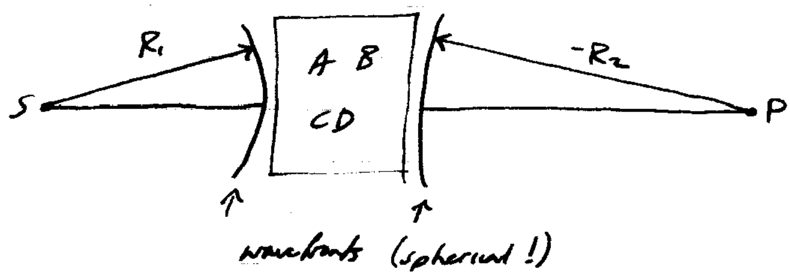
$$y(z)=y_0\cos\alpha z+\frac{\theta_0}{\alpha}\sin\alpha z$$
$$\theta(z)=-y_0\alpha\cos\alpha z+\theta_0\sin\alpha z$$

⇨ $\overleftrightarrow{M}=\begin{bmatrix}\cos\alpha z & \frac{1}{\alpha}\sin\alpha z\\ -\alpha\sin\alpha z & \cos\alpha z\end{bmatrix}$ for propagation our distance z

<u>Transformation of wavefronts</u>

We know that rays ⊥ wavefronts

⇨ In imaging setup , we have



The ABCD matrix transforms the ray angles and displacements such that S and P are conjugate. It is also interesting (and will turn out to be extremely useful in the context of Gaussian beams!) to consider the effect of the element ABCD on the wavefront.

At a distance R from a point (e.g. 5), a ray's height and angle must be related by

$$y=R\theta$$

⇨ We can write the radius of curvature as $R=\frac{y}{\theta}$

So
$$R_1=\frac{y_1}{\theta_1}$$
$$R_2=\frac{y_2}{\theta_2}=\frac{Ay_1+B\theta_1}{Cy_1+D\theta_1}$$

Dividing through by θ_1,

$$R_2 = \frac{A\left(\frac{y_1}{\theta_1}\right) + B}{C\left(\frac{y_1}{\theta_1}\right) + D} \qquad \text{but } \frac{y_1}{\theta_1} = R_1$$

⇨ $$\boxed{R_2 = \frac{A R_1 + B}{C R_1 + D}}$$

This is a perfectly general relationship. Any ABCD system will transform the wavefront radii according to this equation.

<u>Other properties</u> of ABCD matrices:

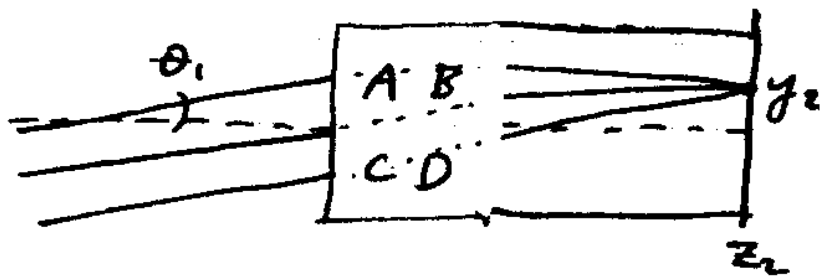
We have seen that B=0 => the ABCD matrix describes propagation between conjugate planes (i.e. plane 1= object, plane 2= image).Now let's consider the other elements:

(i) Suppose A=0.

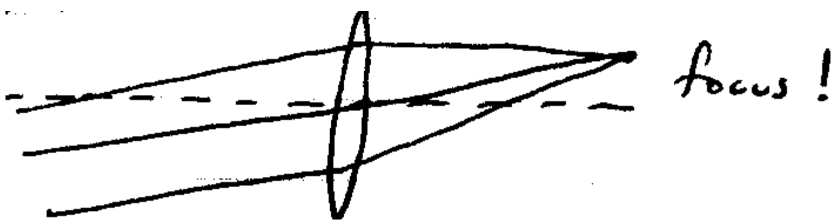
$$\begin{bmatrix} y_2 \\ \theta_2 \end{bmatrix} = \begin{bmatrix} O & B \\ C & D \end{bmatrix} \begin{bmatrix} y_1 \\ \theta_1 \end{bmatrix} \Rightarrow \underline{y_2 = B\theta_1}$$

⇨ All rays that enter the system at the same angle will refract to the same point

Graphically



Recall simple lens

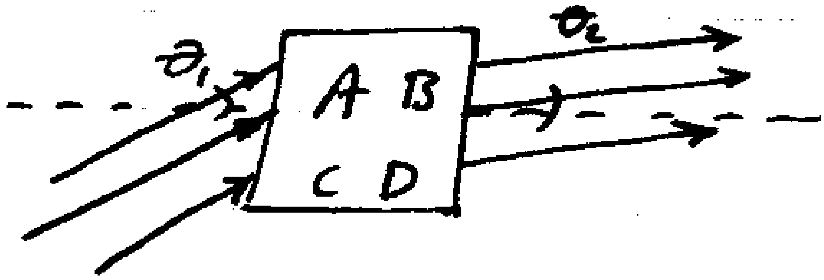


Thus if <u>A=0</u>, the second reference plane of the ABCD system must be the <u>second focal plane</u> of the system!

(ii) Suppose C=0

$$\begin{bmatrix} y_2 \\ \theta_2 \end{bmatrix} = \begin{bmatrix} O & B \\ C & D \end{bmatrix} \begin{bmatrix} y_1 \\ \theta_1 \end{bmatrix} \Rightarrow \theta_2 = D\theta_1$$

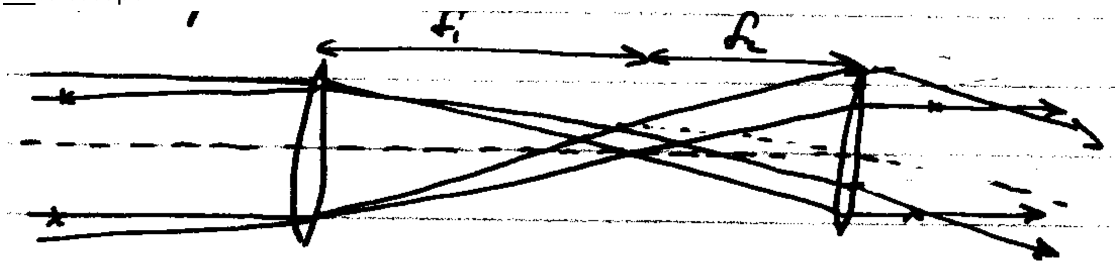
⇨ All parallel input rays are parallel at the output



D= <u>angular magnification</u>

- Such a system is called an <u>afocal</u> or <u>telescopic</u> system (object and image are both at infinity)

<u>Ex</u> telescope

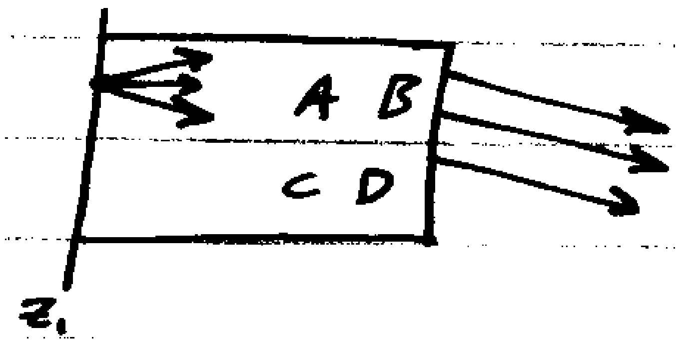


D= 1 => magnification

(iii) Suppose D=0

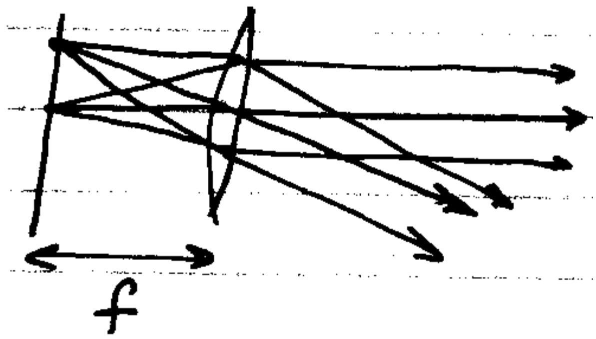
$$\begin{bmatrix} y_2 \\ \theta_2 \end{bmatrix} = \begin{bmatrix} A & B \\ C & D \end{bmatrix} \begin{bmatrix} y_1 \\ \theta_1 \end{bmatrix} \Rightarrow \theta_2 = Cy_1$$

⇨ All rays from a given point y , come out parallel (I.e. at same θ_2)



Thus z_1 must be the front focal plane of the ABCD system.

<u>Ex</u>. simple lens



$$\begin{bmatrix} 1 & 0 \\ -\frac{1}{f} & 1 \end{bmatrix} \begin{bmatrix} 1 & f \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & f \\ -\frac{1}{f} & 1 \end{bmatrix}$$

(iv) General meaning of C :

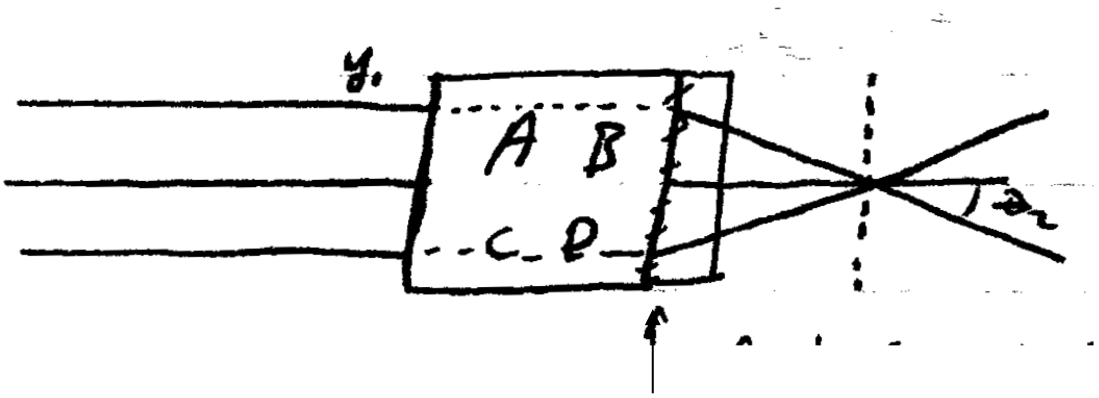
- It is easy to verify for all the above cases that C= $-\frac{1}{f}$.
- This turns out to be generally true ,as is easily seen by considering collimated (parallel) rays at the input of an ABCD system:

$$S \to -\infty, \theta_1 = 0$$

$$\begin{bmatrix} y_2 \\ \theta_2 \end{bmatrix} = \begin{bmatrix} A & B \\ C & D \end{bmatrix} \begin{bmatrix} y_1 \\ 0 \end{bmatrix} \Rightarrow \theta_2 = Cy_1$$

- i.e. output angle is proportional to ray height

⇨ in paraxial approximation , all the rays pass through the same point



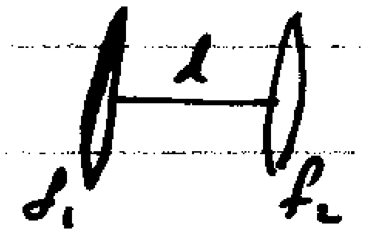
Refraction somewhere inside
(see further notes P.225.5)

Note $\theta_2 < 0$ if C<0 and $y_1 > 0 \Rightarrow$ real image

$$\theta_2 = -\frac{y_1}{f} \quad \text{so} \quad f = -\frac{1}{c}$$

- this provides a convenient way to calculate the focal length of any optical system : find the overall ABCD matrix and then $f = -\frac{1}{c}$.

<u>Example</u>: 2 thin lenses separated by d

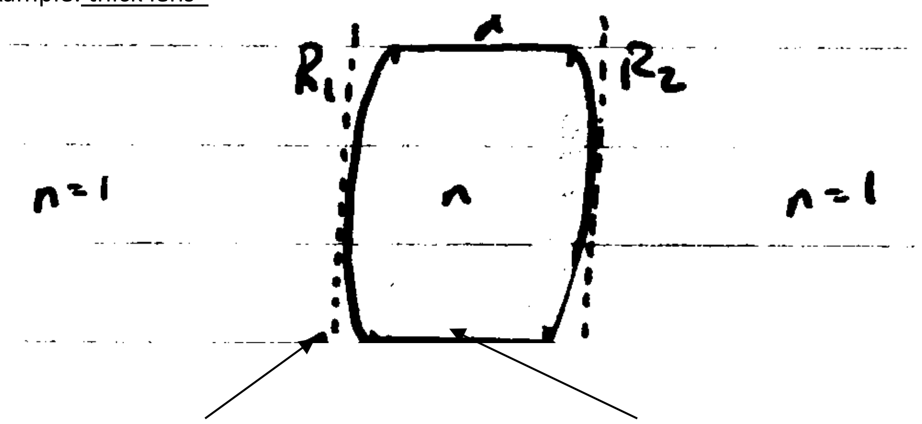


$$\overleftrightarrow{M} = \begin{bmatrix} 1 & 0 \\ -\frac{1}{f_2} & 1 \end{bmatrix} \begin{bmatrix} 1 & d \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ -\frac{1}{f_1} & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ -\frac{1}{f_2} & 1 \end{bmatrix} \begin{bmatrix} 1-\frac{d}{f_1} & d \\ -\frac{1}{f_1} & 1 \end{bmatrix} = \begin{bmatrix} 1-\frac{d}{f_1} & d \\ -\frac{1}{f_2}+-\frac{d}{f_2 f_1}-\frac{1}{f_1} & 1-\frac{d}{f_2} \end{bmatrix}$$

=> $C = -\frac{1}{f} = -\frac{1}{f_1} - \frac{1}{f_2} + \frac{d}{f_2 f_1}$

OR $\boxed{\frac{1}{f} = \frac{1}{f_1} + \frac{1}{f_2} - \frac{d}{f_2 f_1}}$

Example: <u>thick lens</u>



refraction at spherical interface propagation over distance d

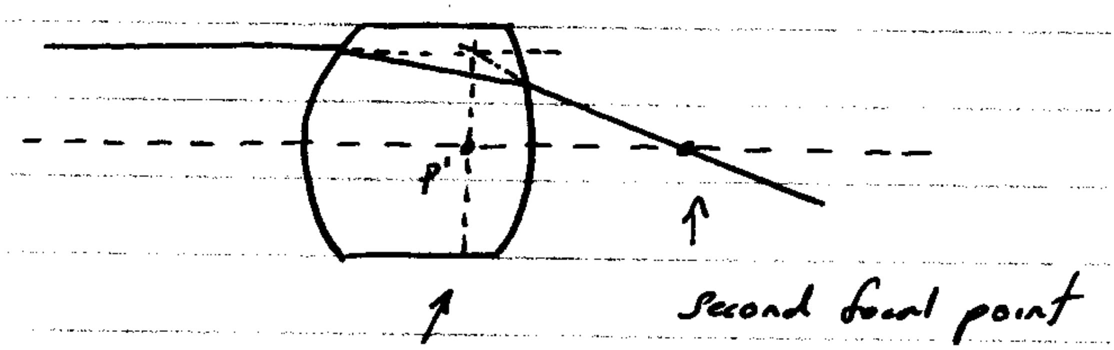
$$\overleftrightarrow{M} = \begin{bmatrix} 1 & 0 \\ -\frac{1-n}{R_2} & 1 \end{bmatrix} \begin{bmatrix} 1 & d \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ -\frac{n-1}{nR_1} & \frac{1}{n} \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ -\frac{1-n}{R_2} & n \end{bmatrix} \begin{bmatrix} 1+d\frac{1-n}{nR_1} & \frac{d}{n} \\ \frac{1-n}{nR_1} & \frac{1}{n} \end{bmatrix} = \begin{bmatrix} 1+d\frac{1-n}{nR_1} & \frac{d}{n} \\ -\frac{1-n}{nR_2}(1+d\frac{1-n}{nR_1})+\frac{1-n}{R_1} & 1-d\frac{1-n}{nR_2} \end{bmatrix}$$

$$C = -\frac{1}{f} = \frac{1-n}{R_1} - \frac{1-n}{R_2} - \frac{(1-n)^2 d}{nR_1R_2}$$

$$\boxed{\frac{1}{f} = (n-1)\left[\left(\frac{1}{R_1} - \frac{1}{R_2}\right) + \frac{(1-n)d}{nR_1R_2}\right]}$$

focal length of a thick lens

Consider now the propagation of an input ray parallel to the optics axis; suppose you applied Snell's law at each interface (e.g. in a ray-tracing program):

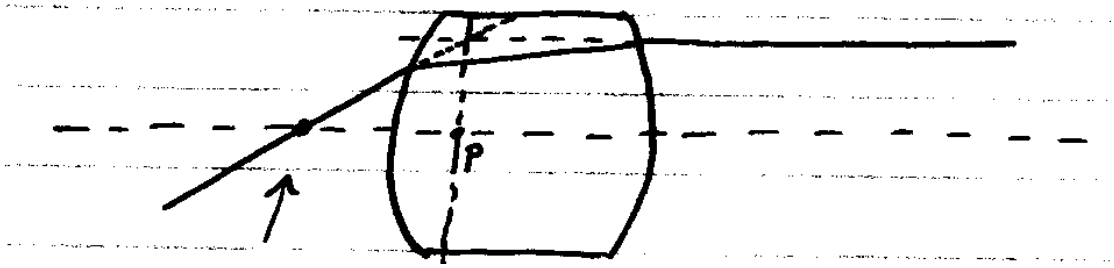


- The refraction looked like it happened here!
- i.e. the thick lens behaved like a thin lens located in the plane containing P'

P' = "secondary principal point"

Plane ="secondary principal plane"

Similarly, we can trace a ray from the primary (first) focal point:



First focal point

"<u>primary principal plane</u>"

P="primary principal point "

(note: clearly the primary and secondary principle planes coincide for a <u>thin</u> lens.)

<u>Significance</u> of <u>principal planes</u>:

You can still use the thin-lens formula

$$\frac{1}{S'}-\frac{1}{S}=\frac{1}{f}$$

as long as you measure S and S' with respect to the primary and secondary principal planes, respectively.

This can be seen by going back to the wavefront transformation eqn. (P.215)

$$R_2=\frac{AR_1+B}{CR_1+D}$$

A little uninteresting algebra can be used to write this in the form

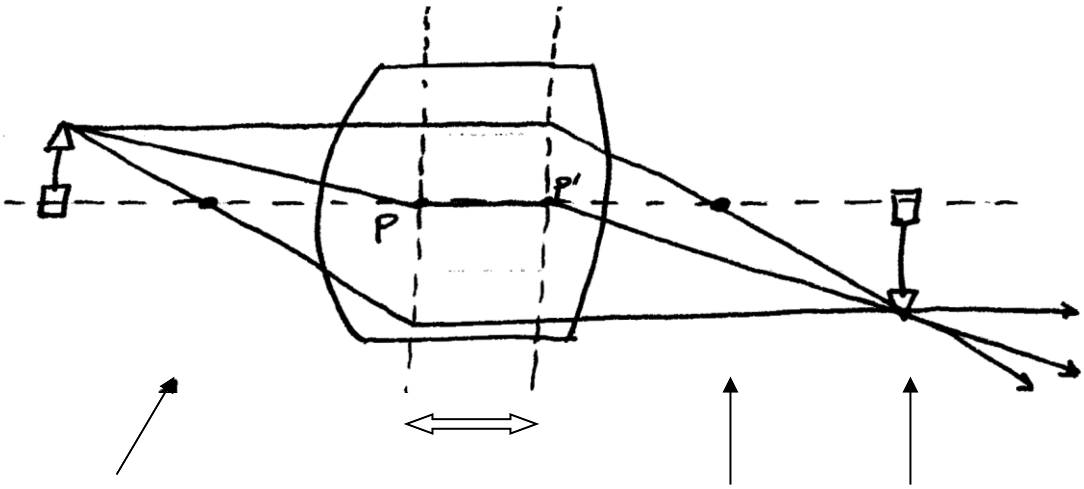
$$\frac{1}{R_2+h_2}-\frac{1}{R_1+h_1}=C=-\frac{1}{f}$$

Where $h_1=\frac{D-1}{C}, h_2=\frac{1-A}{C}$

This looks just like the thin lens equation, but with the object and image points measured with respect to the principal planes (i.e. displaced by h_1 and h_2).

Graphically:

Trace rays exactly as for a thin lens (i.e. the same three rays), but the rays propagate parallel to the axis between the principal planes.



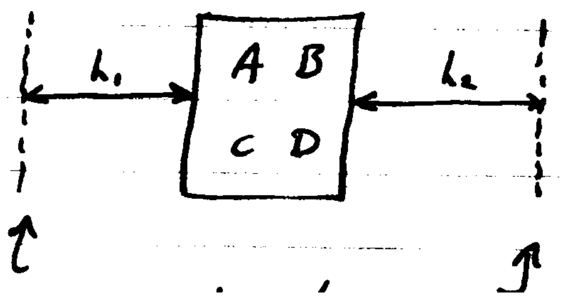
note: rays through principal points P and p' are still parallel outside the lens (as long as n is the same on both sides , as we have been assuming).

Note also that since the rays are parallel between the principle planes, the "magnification" is unify, so the matrix must be of the form.

$$\begin{bmatrix} 1 & \theta' \\ -\frac{1}{f} & 1 \end{bmatrix}$$

Between the two principal planes .

This actually gives another way of calculating the positions of the principal planes for an arbitrary ABCD system:



Principal planes

$$M=\begin{bmatrix}1 & h_2\\0 & 1\end{bmatrix}\begin{bmatrix}A & B\\C & D\end{bmatrix}\begin{bmatrix}1 & -h_1\\0 & 1\end{bmatrix}=\begin{bmatrix}1 & h_2\\0 & 1\end{bmatrix}\begin{bmatrix}A & B-Ah_1\\C & D-Ch_1\end{bmatrix}$$

$$=\begin{bmatrix}A+Ch_2 & B-Ah_1+h_2(D-Ch_1)\\C & D-Ch_1\end{bmatrix}=\begin{bmatrix}1 & B'\\-\frac{1}{f} & 1\end{bmatrix}$$ required

$$A+Ch_2=1\Rightarrow \boxed{h_2=\frac{1-A}{C}}$$

$$D-Ch_1=1\Rightarrow \boxed{h_1=\frac{D-1}{C}}$$

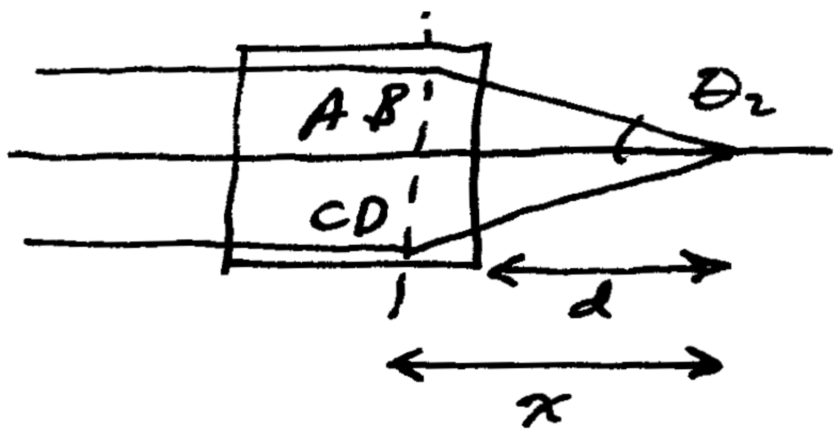
The general positions of the principal +focal planes are shown in Siegman fig.15.12 (be careful about Guenther vs. Siegman Sign Conventions!)

Let's see how this works with an example.

Consider an object at -∞ on axis, so $\theta_1=0$

Q1: at what distance d behind the back reference plane of the ABCD system do the rays come to a focus?

Q2: where is 2^{nd} principal plane—let's call it a distance x from the focal point?



A: consider propagation a distance d from the 2^{nd} reference plane:

$$\begin{bmatrix}1 & d\\0 & 1\end{bmatrix}\begin{bmatrix}A & B\\C & D\end{bmatrix}=\begin{bmatrix}A+dC & B+dD\\C & D\end{bmatrix}$$

For $\theta_1=0, y_2=(A+dC)y_1$

At focus, $y_2=0\Rightarrow A+dC=0\Rightarrow d=-\frac{A}{C}$

Now, the angle the rays converge on the focal point is $\theta_2=Cy_1$, when $\theta_1=0$

Now $\tan\theta_2 \simeq \theta_2 = -\frac{y_1}{x}$

$$x = -\frac{y_1}{\theta_2} = -\frac{y_1}{Cy_1} = -\frac{1}{C}$$

Therefore, the rays look like they refract at the 2^{nd} principal plane where

$$\mathrm{h}_2 = d - x = -\frac{A}{C} + \frac{1}{C}$$

$\boxed{\mathrm{h}_2 = \frac{1-A}{C}}$ as it should be