1 Representation of short optical pulses

There are many ways to mathematically represent optical pulses. It is usually quite straightforward, though some subtleties may enter when the pulse is only a few cycles long. For the sake of establishing a systematic notation and terminology, we will give some important representations here.

Simpliest case: amplitude-modulated sine wave

e.g. Gaussian modulation

$$\varepsilon(t) = \varepsilon_0 e^{-at^2} \cos\omega_0 t = \varepsilon_0 e^{-t^2/2\tau^2} \cos\omega_0 t$$

$$= \varepsilon_0 e^{-[(4\ln 2)(t/\tau_p)^2]} \cos\omega_0 t$$

$e^{-[(4\ln 2)(t/\tau_p)^2]}$: Gaussian envelope; $\cos\omega_0 t$: carrier wave.

Why Gaussians?

(1) math is straightforward

(2) commonly produced by actively mode-locked lasers, as we shall see

Complex notation (following Siegman):

$$\varepsilon(t) = Re[\tilde{\varepsilon}(t)] = \frac{1}{2}[\tilde{\varepsilon}(t) + \tilde{\varepsilon}^*(t)] = \frac{1}{2}\tilde{\varepsilon}(t) + c.c.$$

1.1 Irradiance

Irradiance(W/m^2) - usually called the 'intensity' in casual conversation, although it is not correct radiametric usage:

$$I(t) = \epsilon_0 cn \cdot \frac{1}{T} \int_{t-\frac{T}{2}}^{t+\frac{T}{2}} \varepsilon^2(t)dt$$

$\frac{1}{T}\int_{t-\frac{T}{2}}^{t+\frac{T}{2}} \varepsilon^2(t)dt$: average over 1 optical cycle.

e.g. Gaussian

$$I(t) = \epsilon_0 \varepsilon_0^2 cn \cdot \frac{1}{T} \int\limits_{t-\frac{T}{2}}^{t+\frac{T}{2}} e^{-2at^2} \cos^2 \omega_0 t dt$$

In principle you have to stop here and calculate directly. However, in nearly all cases, one can make the **Slowly varying envelop approximation**.

1.2 Slowly varying envelop approximation(SVEA)

The ampliitude of the envelop does not change appreciably over an optical cycle($\tau^{-1} \ll \omega$)

$$I(t) = \epsilon_0 \varepsilon_0^2 cne^{-2at^2} \cdot \frac{1}{T} \int\limits_{t-\frac{T}{2}}^{t+\frac{T}{2}} \cos^2 \omega_0 t dt$$

Thus the irradiance just follows the square of the amplitude envelope function.

$$I(t) = \frac{1}{2} \epsilon_0 \varepsilon_0^2 cne^{-2at^2} \ (W/m^2)$$

Note that for visible wavelengths $\tau \sim 2fs$

Records(SVEA breaks down!):

1. shortest optical pulse = 2.8 fs

2. shortest directly?? laser = 5 fs

Routine: perhaps $\tau \geq 50fs$ (SVEA is ok)

See Figure 1. FWHM= 1.67τ (Note: breakdown of SVEA in figure)

1.3 complex notation

Usually for shorthand we will of course write

$$\tilde{\varepsilon}(t) = \varepsilon_0 e^{-at^2} e^{i\omega_0 t}$$

and we know that is understood.

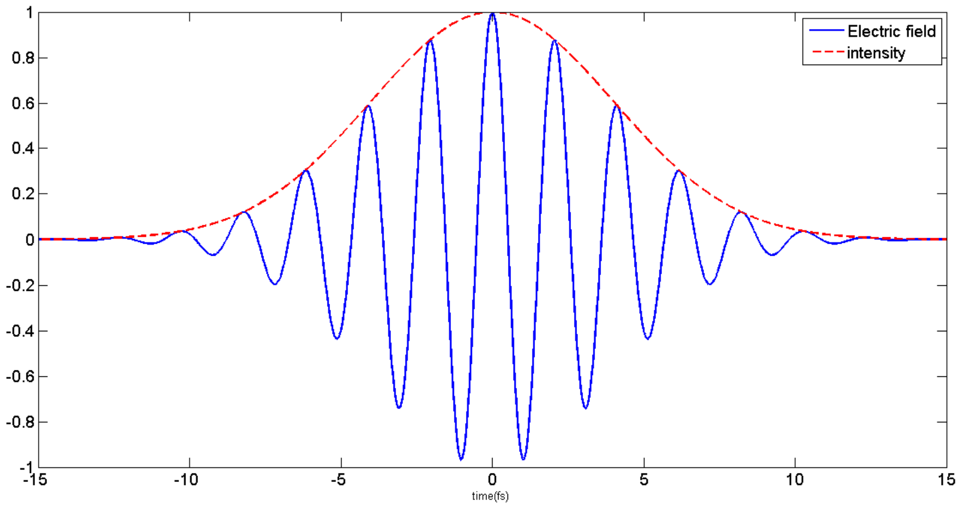

Figure 1: Profile of the electric field of a 6 fs duration pulse at 620 nm (continuous). The broken line curve represents the corresponding intensity profile as a function of time.

Then

$$I(t) = \epsilon_0 cn \cdot \frac{1}{T} \int_{t-\frac{T}{2}}^{t+\frac{T}{2}} [Re(\tilde{\varepsilon})]^2 dt$$

$$= \epsilon_0 cn \cdot \frac{1}{T} \int_{t-\frac{T}{2}}^{t+\frac{T}{2}} [\frac{1}{2}(\tilde{\varepsilon} + \tilde{\varepsilon}^*)]^2 dt$$

$$= \frac{\epsilon_0 cn}{4} \cdot \frac{1}{T} \int_{t-\frac{T}{2}}^{t+\frac{T}{2}} (\tilde{\varepsilon}^2 + 2\tilde{\varepsilon}\tilde{\varepsilon}^* + \tilde{\varepsilon}^{*2}) dt$$

$$= \frac{\epsilon_0 cn}{4} \varepsilon_0^2 e^{-2at^2} \cdot \frac{1}{T} \int_{t-\frac{T}{2}}^{t+\frac{T}{2}} (e^{2i\omega_0 t} + 2 + e^{-2i\omega_0 t}) dt$$

$$(using\, the\, SVEA)$$

As usual, the averages over the period T of the oscillating functions is zero:

$$\frac{1}{T}\int_{t-\frac{T}{2}}^{t+\frac{T}{2}} e^{2i\omega_0 t}dt = \frac{1}{2i\omega_0 T}e^{2i\omega_0 t}|_{t-\frac{T}{2}}^{t+\frac{T}{2}} \qquad (T = \frac{1}{\nu_0} = \frac{2\pi}{\omega_0})$$

$$= \frac{1}{4\pi i}e^{2i\omega_0 t}(e^{2\pi i} - e^{-2\pi i}) = 0$$

the only remaining term is the constant one

$$I(t) = \frac{\epsilon_0 cn}{2}\varepsilon_0^2 e^{-2at^2}$$

which is (of course!) the same result as before.

The real point of this exercise: if you write $\tilde{\varepsilon}(t) = \varepsilon_0 e^{-at^2} e^{i\omega_0 t}$, then the irradiance is

$$I(t) = \frac{1}{2}\epsilon_0 cn \cdot \frac{1}{T}\int_{t-\frac{T}{2}}^{t+\frac{T}{2}} \tilde{\varepsilon}^* \tilde{\varepsilon} dt$$

$$(note\, the\, factor\, \frac{1}{2})$$

1.4 Common pulse shapes + notations

(τ_p= intensity FWHM)

1. Gaussian $\varepsilon(t) \propto e^{-1.386(t/\tau_p)^2} = e^{-(t/\tau)^2}$, $\tau = \frac{\tau_p}{\sqrt{2\ln 2}}$
2. Lorentzian $\varepsilon(t) \propto \frac{1}{1+1.656(t/\tau_p)^2} = \frac{1}{1+(t/\tau)^2}$, (homework)
3. sech $\varepsilon(t) \propto sech\,[1.763t/\tau_p] = sech\,(t/\tau)$, $\tau = \frac{\tau_p}{1.76}$

1.5 Fluence

Fluence is energy per unit area in one pulse:

$$U = \int_{-\infty}^{+\infty} I(t)dt \quad (J/m^2)$$

1.6 Beyond the simplest case

(i) envelope could be arbitrary (not necessarily a simple function!)

(ii) phase advance may not be linear with time

i.e. we had $\tilde{\varepsilon}(t) = E_0 e^{-at^2} e^{i\omega_0 t} = E_0 e^{-at^2} e^{i\Phi(t)}$

where $\Phi(t) = \omega_0 t$, and phase is linear in time.

Taking (i) and (ii) more generally, we may have

$$\tilde{\varepsilon}(t) = E(t) e^{i\Phi(t)}$$

$E(t)$: envelope(usually slowly varing); $\Phi(t)$: phase.

1.7 Example(extremely common): quadratic temporal phase

$$\Phi_{tot}(t) = \omega_0 t + bt^2$$

The 'instantaneous frequency' is the rate of phase advance:

$$\omega \equiv \frac{d\Phi_{tot}}{dt}$$

$$\omega = \frac{d}{dt}(\omega_0 t + bt^2) = \omega_0 + 2bt$$

Thus the frequency changes with time, called a **chirp** (in this case a linear chirp).

example: linearly chirped Gaussian pulse (Figure 2)

$$\tilde{\varepsilon}(t) = E_0 e^{-at^2} e^{i(\omega_0 t + bt^2)}$$

$$= E_0 e^{-(a-ib)t^2} e^{i\omega_0 t}$$

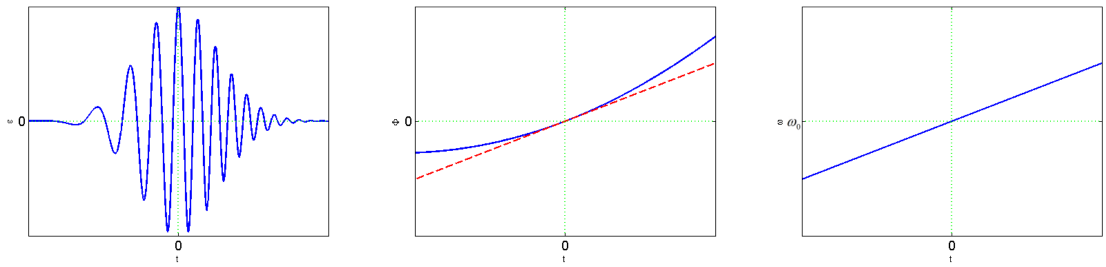

Figure 2: linearly chirped Gaussian pulse. Red dashed curve: $\Phi(t) = \omega_0 t$

$$= E_0 e^{-\Gamma t^2} e^{i\omega_0 t}$$

Thus we have written the pulse as a product of a slowly varying amplitude **and** phase, and an optical (rapidly oscillating) carrier wave. This is useful only if both a and b are small, so that both the intensity envelope and the nonlinear phase vary slowly relative to the carrier frequency. (Clearly the wave drawn at the top does not satisfy that condition!)

Irradiance

$$I_0(t) \propto \tilde{\varepsilon}^* \tilde{\varepsilon}$$

$$= I_0 e^{-2at^2}$$

$$= I_0 e^{-[(4\ln 2)(t/\tau_p)^2]}$$

which depends only on the a parameter. $\tau_p = \sqrt{\frac{2\ln 2}{a}} = FWHM$

1.8 Gaussian pulse spectrum

Of course, the field in the frequency domain is related to the field in the time domain via the **Fourier transform:**

$$\tilde{\varepsilon}(\omega) = \int\limits_{-\infty}^{+\infty} \tilde{\varepsilon}(t) e^{-i\omega t} dt$$

$$= E_0 \int\limits_{-\infty}^{+\infty} e^{-\Gamma t^2} e^{i(\omega_0 - \omega)t} dt$$

This integral may be evaluated by completing the square in the exponent:

$$\tilde{\varepsilon}(\omega) \sim \int\limits_{-\infty}^{+\infty} e^{-\Gamma t^2 - 2Bt} dt \qquad 2B = -i(\omega_0 - \omega)$$

$$= \int\limits_{-\infty}^{+\infty} e^{-\Gamma(t^2 + \frac{2B}{\Gamma}t)} dt = \int\limits_{-\infty}^{+\infty} e^{-\Gamma[(t + \frac{B}{\Gamma})^2 - \frac{B^2}{\Gamma^2}]} dt$$

$$= e^{\frac{B^2}{\Gamma}} \int\limits_{-\infty}^{+\infty} e^{-\Gamma[(t + \frac{B}{\Gamma})^2} dt = e^{\frac{B^2}{\Gamma}} \int\limits_{-\infty}^{+\infty} e^{-\Gamma t'^2} dt'$$

$$= \sqrt{\frac{\pi}{\Gamma}} e^{\frac{B^2}{\Gamma}} \qquad ('Siegman's\, lemma,\, p.337)$$

Thus we find

$$\tilde{\varepsilon}(\omega) = E_0 \sqrt{\frac{\pi}{\Gamma}} e^{-\frac{(\omega_0 - \omega)^2}{4\Gamma}}$$

Or, explicitly in terms of the a and b parameters:

$$\tilde{\varepsilon}(\omega) \sim e^{-\frac{(\omega_0-\omega)^2}{4(a-ib)}}$$

$$= e^{-\frac{1}{4}\frac{(\omega_0-\omega)^2(a+ib)}{a^2+b^2}}$$

$$= e^{-\frac{a}{4}\frac{(\omega_0-\omega)^2}{a^2+b^2}} \times e^{-\frac{ib}{4}\frac{(\omega_0-\omega)^2}{a^2+b^2}}$$

$e^{-\frac{a}{4}\frac{(\omega_0-\omega)^2}{a^2+b^2}}$: Gaussian envelope in frequency domain; $e^{-\frac{ib}{4}\frac{(\omega_0-\omega)^2}{a^2+b^2}}$: quadratic phase shift in frequency domain arising from quadratic shift in time domain ($b \neq 0$)

You must take to your gra?? the knowledge that a '**linear chirp**' corresponds to a **quadratic phase shift** in the frequency domain.