lecture 14
2/9/05

Nonlinear Propagation Equation

We want to extend the parabolic eqn. to cover the presence of a weak, instantaneous, nonresonant Kerr nonlinearity.

As usual, we start with Maxwell's wave eqn.

$$\frac{\partial^2 \mathcal{E}}{\partial z^2} - \frac{1}{c^2}\frac{\partial^2 \mathcal{E}}{\partial t^2} = \mu_0 \frac{\partial^2 P}{\partial t^2} \quad \text{(for a plane wave)}$$

Also as usual, we consider one frequency component

$$\mathcal{E}(\omega) = E(\omega) e^{i\omega t} \qquad P(\omega) = p(\omega) e^{i\omega t} = P_L + P_{NL}$$

$$\frac{\partial^2 \mathcal{E}}{\partial t^2} = -\omega^2 \mathcal{E} \qquad = (p_L + p_{NL}) e^{i\omega t}$$

$$\frac{\partial^2 P_L}{\partial t^2} = -\omega^2 P_L$$

For now we will keep the nonlinear term in the form

$$\frac{\partial^2 P_{NL}}{\partial t^2}.$$

Substituting in the wave equation gives

$$\frac{\partial^2 \mathcal{E}}{\partial z^2} + \frac{\omega^2}{c^2}\mathcal{E} = -\mu_0 \omega^2 P_L + \mu_0 \frac{\partial^2 P_{NL}}{\partial t^2}$$

$$P_L = \epsilon_0 \chi^{(1)} \mathcal{E} \quad \Rightarrow$$

$$\frac{\partial^2 \mathcal{E}}{\partial z^2} + \frac{\omega^2}{c^2}(1+\chi^{(1)})\mathcal{E} = \mu_0 \frac{\partial^2 P_{NL}}{\partial t^2}$$

Now we must consider the $\partial^2 P_{NL}/\partial t^2$ term.

We start with (see p. 166)

$$\mathcal{P}_{NL} = \frac{3}{4}\epsilon_0 \chi^{(3)} |\mathcal{E}|^2 \mathcal{E}$$

$$= \frac{3}{4}\epsilon_0 \chi^{(3)} |E|^2 E e^{i\omega t}$$

(again, we are assuming a real, instantaneous, nonresonant nonlinear response).

The source term in the wave eqn. is then

$$\mu_0 \frac{\partial^2 \mathcal{P}_{NL}}{\partial t^2} = \frac{3\mu_0\epsilon_0}{4}\chi^{(3)} \frac{\partial^2}{\partial t^2}\left\{ |E|^2 E e^{i\omega t} \right\}$$

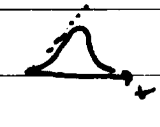
$$= \frac{3\mu_0\epsilon_0}{4}\chi^{(3)} \Big\{ -\omega^2 |E|^2 E$$

$$+ 2i\omega \frac{\partial}{\partial t}\left(|E|^2 E\right) + \frac{\partial^2}{\partial t^2}\left(|E|^2 E\right) \Big\} e^{i\omega t}$$

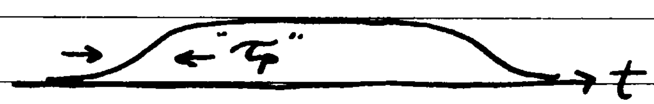
$$\mu_0 \frac{\partial^2 P_{NL}}{\partial t^2} = -\frac{3\mu_0\epsilon_0}{4}\omega^2\chi^{(3)}(\omega) \times \left\{ |E|^2E - \frac{2i}{\omega}\frac{\partial}{\partial t}\left(|E|^2E\right) - \frac{1}{\omega^2}\frac{\partial^2}{\partial t^2}\left(|E|^2E\right)\right\} e^{i\omega t}$$

Note that the order of magnitude of

$\frac{\partial}{\partial t}\left(|E|^2 E\right)$ is approximately $\frac{1}{\tau_p}\left(|E|^2E\right)$,



since $E(t)$ is the pulse slowly varying envelope. (Of course a pulse with a lot of structure on it, such as one might have from a pulse shaper, will satisfy this as τ_p is the time scale of the fastest structure on the pulse, rather than the overall length of the pulse; e.g. for a square pulse, the appropriate τ_p is shown:)



Thus we could define a smallness parameter (or "wave transience parameter")

$$\mu = \frac{T}{\pi\tau_p} = \frac{2}{\omega\tau_p}$$

where T = optical period ($\sim$ 2 fs).

Thus the three terms in the {} brackets are of order

$$\{\mathcal{O}(1) - \mathcal{O}(\mu) - \mathcal{O}(\mu^2)\}.$$

If the pulse is more than a few cycles long, then <u>$\mu << 1$</u>. Thus we will completely neglect the $\mathcal{O}(\mu^2)$ term. For the moment we shall set aside the $\mathcal{O}(\mu)$ term (see p. 182), and consider only the first term.

$$\mu_0 \frac{\partial^2 P_{NL}}{\partial t^2} \simeq -\frac{3}{4} \frac{\omega^2}{c^2} \chi^{(3)}(\omega) |E|^2 E e^{i\omega t}$$

Plugging this back in $\circledast$ p. 171

$$\frac{\partial^2 \mathcal{E}}{\partial z^2} + \left[\frac{\omega^2}{c^2}\left(1+\chi^{(1)}\right) + \frac{\omega^2}{c^2}\left(\frac{3\chi^{(3)}}{4}\right)|E|^2\right]\mathcal{E} = 0$$

$$\frac{\partial^2 \mathcal{E}}{\partial z^2} + \frac{n_0^2 \omega^2}{c^2}\left[1 + \frac{3\chi^{(3)}}{4 n_0^2}|E|^2\right]\mathcal{E} = 0$$

or $$\frac{\partial^2 \mathcal{E}}{\partial z^2} + \beta^2 \mathcal{E} = 0$$

$$n_2 = \frac{3\chi^{(3)}}{8 n_0}$$

where

$$\beta^2 = \frac{n_0^2 \omega^2}{c^2}\left[1 + \frac{2 n_2}{n_0}|E|^2\right]$$

$$\beta \simeq \beta_0\left[1 + \frac{n_2}{n_0}|E|^2\right]$$

$$\beta \simeq \beta_0 + \delta\beta \quad , \quad \delta\beta = \beta_0 \frac{n_2}{n_0}|E|^2$$

Now, our goal is (as it was on p.37 in our derivation of the parabolic propagation equation) to find an equation for the field envelope $E(z,t)$ in

$$\mathcal{E}(z,t) = E(z,t)\, e^{i[\omega_o t - \beta(\omega_o) z]}$$

starting with our wave eqn. $\frac{\partial^2 \mathcal{E}}{\partial z^2} + \beta^2 \mathcal{E} = 0$, but now β is intensity - dependent.

As with our earlier derivation, we start with

$$\mathcal{E}(z,\omega) = \mathcal{E}(0,\omega)\, e^{-i\beta z} \longrightarrow \frac{\partial \mathcal{E}}{\partial z} = -i\beta \mathcal{E}$$

$$\frac{\partial \mathcal{E}}{\partial z} = -i\left\{ \beta(\omega_o) + \beta'(\omega-\omega_o) + \tfrac{1}{2}\beta''(\omega-\omega_o)^2 + \delta\beta \right\}\mathcal{E}$$

$$\frac{\partial E}{\partial z} - i\beta(\omega_o) E = -i\left\{ \beta(\omega_o) + \beta'(\omega-\omega_o) + \tfrac{1}{2}\beta''(\omega-\omega_o)^2 + \delta\beta \right\} E$$

The $\beta(\omega_o)E$ term cancels, and a Fourier transform is carried out just as in our previous derivation to yield

$$\boxed{\frac{\partial E}{\partial z} + \frac{1}{v_g}\frac{\partial E}{\partial t} - \frac{i}{2}\beta''\frac{\partial^2 E}{\partial t^2} + i k |E|^2 E = 0}$$

where

$$\underline{k = \beta(\omega_o)\frac{n_2}{n_o} = \frac{n_2 \omega_o}{c}}.$$

This "<u>nonlinear</u> <u>Schrödinger</u> <u>eqn.</u>" describes the propagation of the field envelope (slowly varying amplitude and phase) in a weakly nonlinear Kerr medium in the presence of GVD.

Note: we assumed an infinite plane wave in order to generate a 1-D equation. That of course isn't physically realistic (since the nonlinearity couples the spatial and temporal propagation). In an optical fiber, however, the <u>single-mode</u> nature of the field profile gaurantees that the 1-D model applies. In that case β is the <u>propagation constant</u> of the <u>fundamental mode</u> in the fiber.

Haus (Optoelectronics section 6.7) has shown that for a single-mode fiber

$$\delta\beta = \frac{\omega^2 \mu_0 \epsilon_0}{\beta} \frac{\int da\, n\, \delta n |u|^2}{\int da\, |u|^2}$$

where

$$\mathcal{E}(x,y,z,t) = \underbrace{E(z,t)}_{\text{propagating envelope}} \; \underbrace{u(x,y)}_{\substack{\text{transverse profile} \\ \text{(Gaussian)}}}$$

$\int da \leftrightarrow$ integral over fiber transverse profile

$$\delta n = \bar{n}_2 I = n_2 |E|^2 |u|^2$$

$$\Rightarrow \delta\beta = \left[\frac{\omega^2 \mu_0 \epsilon_0}{\beta} \frac{\int da\, n_0 n_2 |u|^4}{\int da\, |u|^2} \right] |E|^2 = k |E|^2$$

It is also worth noting the connection between our notation and that of Agrawal's Nonlinear Fiber Optics. Our version of the nonlinear Schrödinger eqn. is written in terms of the field amplitude E. It is perhaps more useful to write a version of the nonlinear Sch. eqn. in terms of the power, which is a more accessible quantity experimentally; this is the approach taken in Agrawal chap. 2.

To transform our NLSE to Agrawal's, we start with the nonlinear term

$$K|E|^2 = \frac{\omega_0}{c} n_2 |E|^2$$

$$= \frac{\omega_0}{c} \bar{n}_2 I$$

$$= \frac{\omega_0}{c} \bar{n}_2 \frac{P}{A_{eff}}$$

where P = (instantaneous) power in the pulse

A_{eff} = effective area of the beam

$$\equiv \frac{\left[\iint dx\,dy\, |u(x,y)|^2\right]^2}{\iint dx\,dy\, |u(x,y)|^4}$$

$= \pi W^2$ for a Gaussian beam (e.g. in single-mode fiber)

The nonlinear term can thus be written

$$\kappa |E|^2 = \frac{\omega_0 \bar{n}_2}{c A_{eff}} P$$

$$= \gamma P \quad , \quad \gamma = \frac{\bar{n}_2 \omega_0}{c A_{eff}}$$

$$= \gamma |A|^2$$

where A = field amplitude scaled such that $|A|^2$ = power in beam.

The NLSE is then

$$\frac{\partial A}{\partial z} + \frac{1}{V_g}\frac{\partial A}{\partial t} - \frac{i}{2}\beta'' \frac{\partial^2 A}{\partial t^2} + i\gamma |A|^2 A = 0$$

(This is Agrawal's eqn. 2.3.27, except for a sign change $i \to -i$. This arises simply because we write $\mathcal{E}(t) \sim e^{i\omega t}$ and he writes $\mathcal{E}(t) \sim e^{-i\omega t}$; there is no physical consequence).

SPM revisited

Consider propagation in a short medium so that GVD can be neglected.

$$\frac{\partial E}{\partial z} + \frac{1}{V_g}\frac{\partial E}{\partial t} + i\kappa|E|^2 E = 0$$

This is even easier to work with in the moving coordinate system

$$\tau = t - \frac{z}{V_g} \quad , \quad \xi = z$$

$$\frac{\partial E}{\partial \xi} + i\kappa|E|^2 E = 0$$

that's $E_0(\tau)$! (i.e. the input pulse)

$\Rightarrow$ solution $E(\xi,\tau) = E_0(\tau)\, e^{-i\kappa|E_0|^2 \xi}$

check : $\frac{\partial E}{\partial \xi} = -i\kappa|E_0|^2 E$

But $|E_0|^2 = |E|^2$ (phase term cancels in taking mod squared)

$\rightarrow$ the pulse shape does not change (in a short medium)

Only the phase changes.

$\Rightarrow$ the equation governing the phase development is

$$\phi_{NL}(\tau,\xi) = -\kappa|E_0(\tau)|^2 \xi$$

Consider propagation of a Gaussian pulse

$$E_0(\tau) = E_0 e^{-\tau^2/2\tau_0^2}$$

Clearly the maximum phase shift will occur at the peak of the pulse ($\tau = 0$) $\Rightarrow$ define

$$|\varphi_{max}| = \kappa |E_0(0)|^2 \xi \qquad (= \gamma P_0 \xi \text{ where } P_0 = \text{peak power})$$

Define the nonlinear SPM length as the distance at which $|\varphi_{max}| = 1$

$$L_{SPM} = \frac{1}{\kappa |E_0(0)|^2} \qquad \left(= \frac{1}{\gamma P_0} \right)$$

As before, we can define the instantaneous frequency as

$$\delta\omega(t) \equiv \frac{d\varphi_{NL}(t)}{dt} = \frac{\partial\varphi_{NL}(\tau)}{\partial\tau} = -\kappa \frac{\partial |E_0|^2}{\partial\tau} \xi$$

The following figure shows plots of the phase and frequency for normalized time.

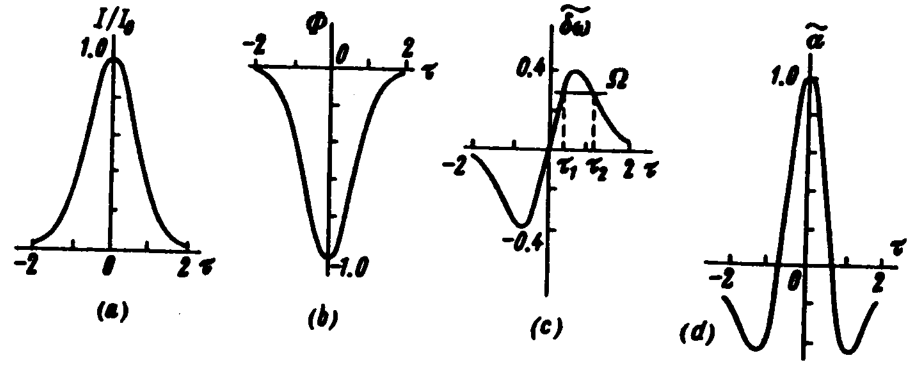


FIG. 2.3. Shape of a Gaussian pulse (a), the reduced phase $\Phi = \varphi/\varphi_{max}$ (b), reduced frequency deviation $\widetilde{\delta\omega}(t) = \delta\omega(t)/\delta\omega_0$ (c), and reduced frequency variation rate $\tilde{\alpha}(t, z) = \alpha(t, z)/\alpha(0, z)$ (d) as a function of time $\tau = \eta/\tau_0$; $\delta\omega_0 = 2\varphi_{max}/\tau_0$, $\alpha(0, z) = 2\varphi_{max}/\tau_0^2$.

SPM spectrum

As we noted earlier, the pulse shape (Gaussian in our example) does not change with propagation, but the pulse spectrum does; it continually broadens with increasing propagation distance z (or ξ).

$$E(\xi,\Omega) = \int E(\xi,\tau)\, e^{-i\Omega\tau}\, d\tau$$

where $\Omega = \omega - \omega_0$.

Power spectrum

$$S(\xi,\Omega) = |E(\xi,\Omega)|^2 = \left|\int E_0(\tau)\, e^{-i[\Omega\tau - \varphi(\xi,\tau)]}\, d\tau\right|^2.$$

Obviously, analytical solutions to this Fourier transform are difficult: some numerical solutions are shown on p. 180. Agreement with experimental data is quite good.

Note

(1) severe ~~pulse~~ spectral distortion begins to appear for a max. phase shift of $\sim \pi$.

(2) The spectrum not only gets broader with propagation; the energy moves out from the central frequency into the wings. Since the pulse shape doesn't change, the front end continually red shifts and the back end continually blue shifts. Structure appears in the spectrum when $\varphi_{max} > \pi$.

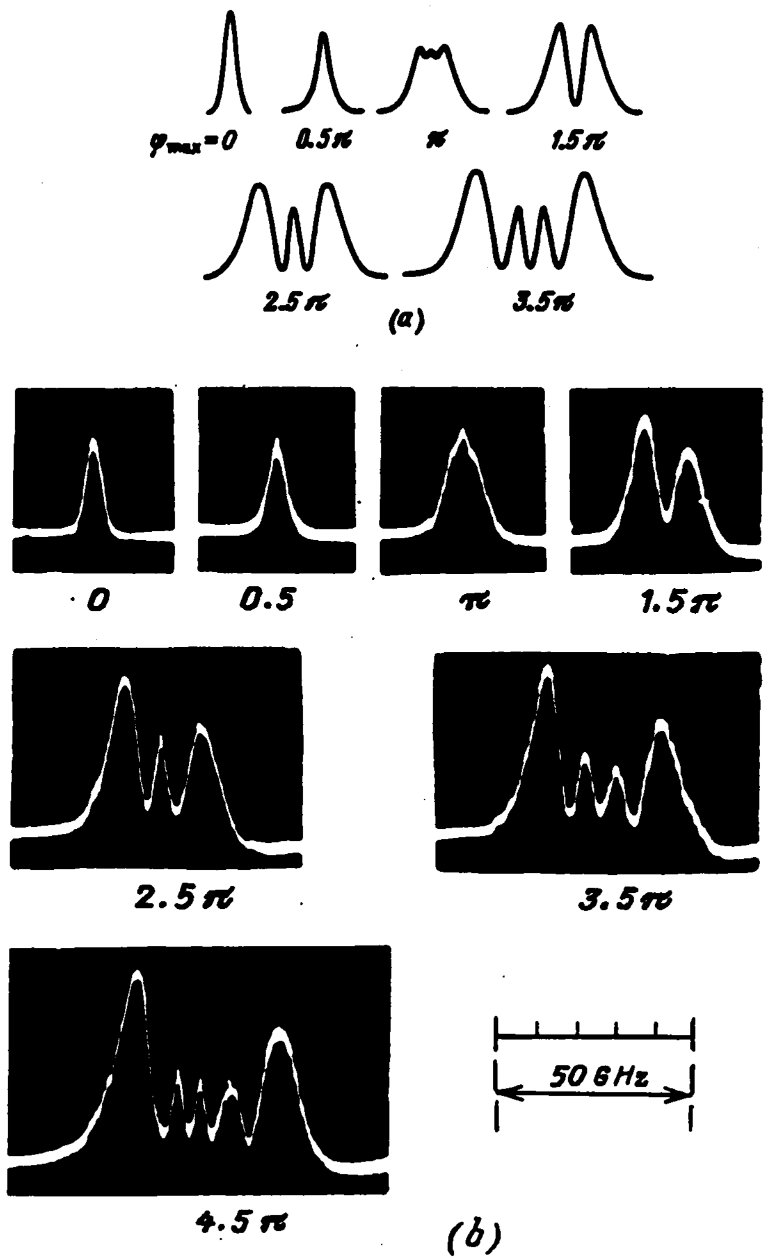


FIG. 2.4. Spectrum of a Gaussian pulse for different maximum values of phase φ_{max} : (a) theory; (b) experiment.[4]

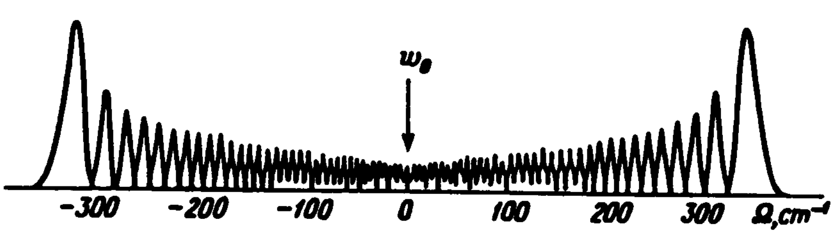


FIG. 2.5. Spectrum of a Gaussian pulse after SPM for $\varphi_{max} \gg 1$.[5]

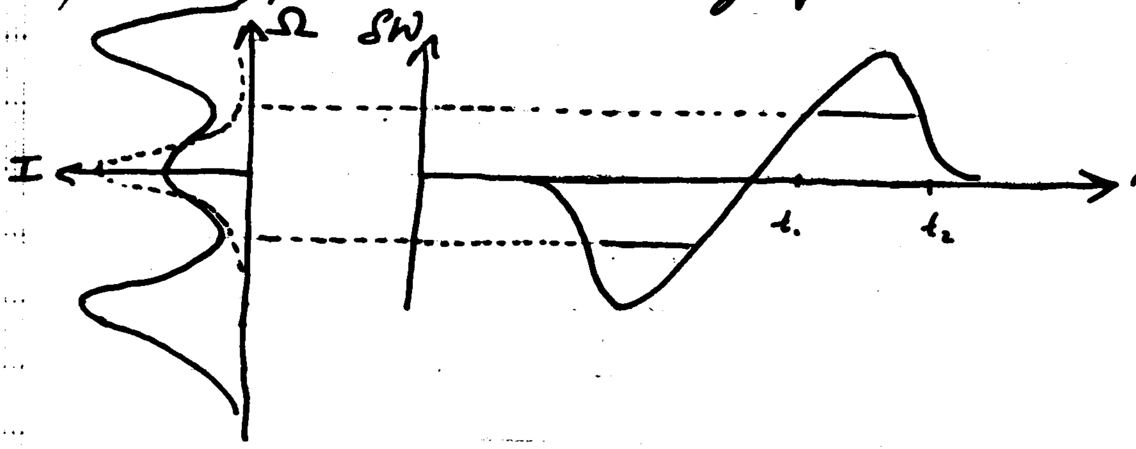
Parameters in the simulation with pure SPM

$\gamma = 3W^{-1}km^{-1}$ **(typical value for SMF)**

Pulse duration = 100 fs

Peak power = 1 kW

The structure in the spectrum can be understood qualitatively from the following picture:



When the phase shift is large, so that the frequency components δW at t_1 and t_2 are out of phase by π, 3π, 5π, ..., then destructive interference will ~~give~~, rise to a minimum in the spectrum at

$$\Omega = \delta W(t_1) \text{ or } \delta W(t_2) .$$

This may be understood by a "stationary phase" requirement: when $\varphi_{max} >> 1$, the largest contributions to the Fourier transform integral occur when

$$\frac{\partial}{\partial \tau}\left[\Omega\tau - \varphi(z,\tau)\right] = 0$$

$$\Omega - \frac{\partial\varphi}{\partial\tau} = 0$$

$$\Omega - \delta W = 0 .$$

This relation does not hold when φ_{max} is not very large (i.e. is only a few $\times\, \pi$).

(3) the ideal spectrum is symmetric for a temporally symmetric pulse. An asymmetric pulse will produce an asymmetric spectrum.

(4) for a Gaussian pulse the maximum shift can be shown to be

$$\delta\omega_{max} = 0.43\, \varphi_{max}\, \Delta\omega_0 \qquad \left(\Delta\omega_0 = \frac{2}{\tau_0}\right)$$

so that the bandwidth is

$$\boxed{\Delta\omega' = 0.86\, \varphi_{max}\, \Delta\omega_0}$$

(5) the total number of maxima in the spectrum is the integer part of φ_{max}/π.

(6) It is important to note that this strong interference structure only appears when there is strong SPM <u>without</u> <u>GVD</u>. For propagation of high energy pulses through short dielectrics, this structure can be observed. For <u>low-energy</u> pulses (e.g. from mode-locked laser oscillators), phase shifts of $> \pi$ require <u>long</u> propagation lengths since $n_2 I$ is low. In that case GVD is certainly <u>not</u> negligible.