$$\tilde{E}(z,\omega) = e^{-i\beta(\omega_0)z - i\beta'(\omega-\omega_0)z - (\frac{1}{4\Gamma_0} + i\frac{1}{2}\beta''z)(\omega-\omega_0)^2}$$

It is worth reiterating that, since this is linear propagation, the output pulse spectrum is the same as the input spectrum

$$|\tilde{E}(z,\omega)|^2 = |\tilde{E}_0(\omega)|^2$$

However, the pulse does change in the time domain. According to our program, the output is the inverse Fourier transform

$$\tilde{\varepsilon}(z,t) = \int_{-\infty}^{+\infty} \tilde{E}(z,\omega) e^{i\omega t} d\omega$$

$$= e^{-i\beta(\omega_0)z} \int_{-\infty}^{+\infty} e^{i\omega t - i\beta'(\omega-\omega_0)z - (\frac{1}{4\Gamma_0} + i\frac{1}{2}\beta''z)(\omega-\omega_0)^2} d\omega$$

$$= e^{i[\omega_0 t - \beta(\omega_0)z]} \int_{-\infty}^{+\infty} e^{i(t-\beta'z)(\omega-\omega_0) - (\frac{1}{4\Gamma_0} + i\frac{1}{2}\beta''z)(\omega-\omega_0)^2} d(\omega-\omega_0)$$

define

$$\frac{1}{\Gamma(z)} = \frac{1}{\Gamma_0} + 2i\beta''z$$

$$(envelope\, and\, chirp\, parameters\, are\, spatially\, varying)$$

$$\tilde{\varepsilon}(z,t) = e^{i[\omega_0 t - \beta(\omega_0)z]} \int_{-\infty}^{+\infty} e^{i(t-\beta'z)(\omega-\omega_0) - \frac{1}{4\Gamma(z)}(\omega-\omega_0)^2} d(\omega-\omega_0)$$

$e^{i[\omega_0 t - \beta(\omega_0)z]}$: time and space dependence of carrier wave; $e^{i(t-\beta'z)(\omega-\omega_0) - \frac{1}{4\Gamma(z)}(\omega-\omega_0)^2}$: time and space dependence of (slowly varying) envelope.

It is clear on inspection that the temporal form of the envelope is $e^{-\Gamma t^2}$, but with a shift in time of $t - \beta' z$.

Calculating explicitly (using 'Siegman's lemma' again):

$$\tilde{\varepsilon}(z,t) = e^{i[\omega_0 t - \beta(\omega_0)z]} e^{-\Gamma(z)(t-\beta'z)^2}$$

$$= e^{i\omega_0[t-\frac{\beta(\omega_0)z}{\omega_0}]} \times e^{-\Gamma(z)(t-\beta' z)^2}$$

$$= e^{i\omega_0[t-\frac{z}{v_\Phi(\omega_0)}]} \times e^{-\Gamma(z)(t-\frac{z}{v_g(\omega_0)})^2}$$

Thus after propagation over a distance z, there is a **phase delay** $t_\Phi = z/v_\Phi(\omega_0)$.

$\Longrightarrow$

$$v_\Phi(\omega_0) = \frac{\omega_0}{\beta(\omega_0)} \qquad '\mathbf{phase\,velocity}'\,at\,the\,carrier\,frequency$$

After a distance z, the peak at the pulse is delayed by a 'group delay' $t_g = z/v_g(\omega_0)$

$\Longrightarrow$

$$v_g(\omega_0) = \frac{1}{\beta'(\omega_0)} = \frac{1}{\frac{d\beta}{d\omega}|_{\omega_0}} = \frac{d\omega}{d\beta}|_{\omega_0}$$

$$'\mathbf{group\,velocity}'\,of\,the\,pulse\,envelope\,at\,the\,carrier\,frequency$$

Note on notation: many texts and publications use k instead of β (and often call it the 'wavevector' instead of the 'propagation constant'). Hence the often quoted

$$v_\Phi(\omega_0) = \frac{\omega}{k}, \quad v_g(\omega_0) = \frac{d\omega}{dk}$$

The utility of the β notation is that it emphasizes that, in waveguide problems, the phase and group velocities are determined by the waveguide eigenvalue $\beta(\omega)$.

A physical picture of GV(group velocity): suppose you are surfing along a wave so that you are in the **rest frame** of the wave **group.** (i.e. going at v_g). Then if $n(\omega)$= constant independent of ω, $v_\Phi = v_g$, and the arrier wave moves along with you.

However, if $n(\omega) \neq$ constant, then $v_\Phi \neq v_g$, and the carrier wave appears to 'slip' in the positive or negative direction according to whether $v_\Phi > v_g$ or $v_\Phi < v_g$.(Figure 1)

Why should it be the **slope** which determines the velocity of the pulse **envelope** (i.e. the wave **group**) ?

Consider the Fourier construction of a pulse: peak is where all the waves add **in phase.**(Figure 2) Peak of envelope is where the variation of phase on frequency is zero.

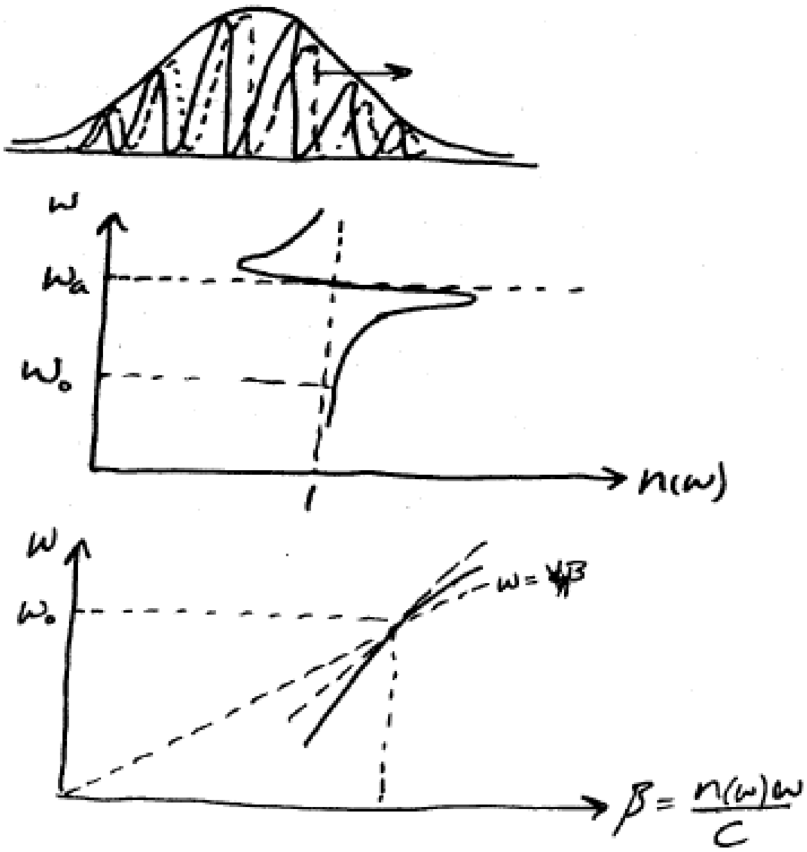

Figure 1: The carrier wave 'slips'.

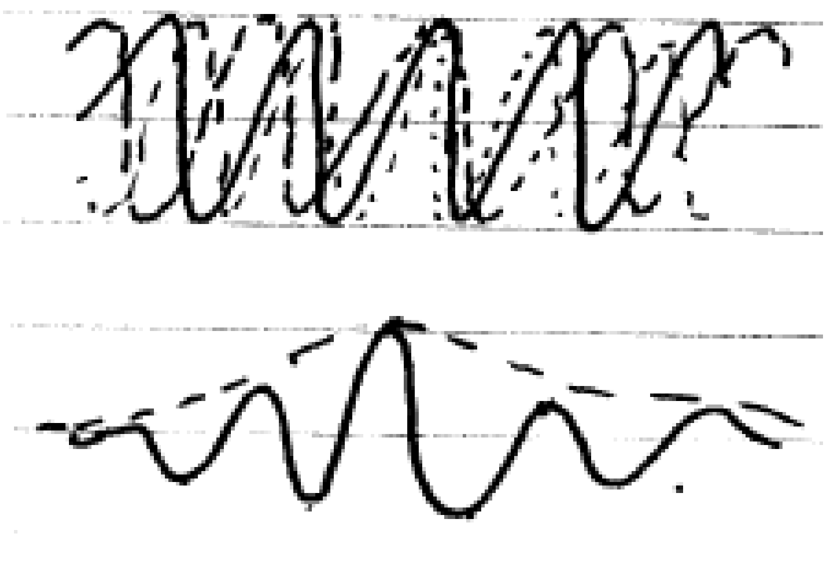

Figure 2: Peak is where all the waves add in phase.

$$\Phi = \omega t - \beta z \Rightarrow \frac{d\Phi}{d\omega} = t - \frac{d\beta}{d\omega} z = 0$$

$$\Rightarrow t = \frac{z}{v_g} \qquad where\ v_g = \frac{d\omega}{d\beta} = group\, velocity$$

Now let us consider the **intensity envelope** in the time domain.

$$I(z,t) \propto |\tilde{\varepsilon}(z,t)|^2 \propto |e^{-\Gamma(z)(t-\frac{z}{v_g})^2}|^2$$

where we must remember that $\Gamma(z)$ is **complex**.

$$\frac{1}{\Gamma(z)} = \frac{1}{\Gamma_0} + 2i\beta'' z$$

$$= \frac{1}{a_0 - ib_0} + 2i\beta'' z = \frac{a_0 + ib_0}{a_0^2 + b_0^2} + 2i\beta'' z$$

$$= \frac{a_0}{a_0^2 + b_0^2} + i(\frac{b_0}{a_0^2 + b_0^2} + 2\beta'' z)$$

Recall that the width in the frequency domain

$$\Delta\omega_p \propto Re[\frac{1}{\Gamma}]$$

$\Longrightarrow$

width is **constant** in frequency domain. (of course! A linear system can add no new frequencies to the propagating field.)

In the time domain, however, we must consider $\Gamma(z)$:

$$\Gamma(z) = \frac{1}{\frac{1}{\Gamma_0} + 2i\beta'' z} = \frac{\Gamma_0}{1 + 2i\beta'' z\Gamma_0}$$

$$= \frac{a_0 - ib_0}{(1 + 2i\beta'' z)(a_0 - ib_0)}$$

$$= \frac{a_0 - ib_0}{1 + 2b_0\beta''z + 2ia_0\beta''z}$$

$$= \frac{(a_0 - ib_0)[1 + 2b_0\beta''z - 2ia_0\beta''z]}{(1 + 2b_0\beta''z)^2 + (2a_0\beta''z)^2}$$

$$= a(z) - ib(z)$$

where

$$a(z) = \frac{a_0}{(1 + 2b_0\beta''z)^2 + (2a_0\beta''z)^2}$$

$$b(z) = \frac{b_0(1 + 2b_0\beta''z) + 2a_0^2\beta''z}{(1 + 2b_0\beta''z)^2 + (2a_0\beta''z)^2}$$

Recall the physical meanings of a and b: a is pulsewidth (of envelope), b is chirp parameter. We find that the pulsewidth (FWHM $\tau_p = \sqrt{\frac{2\ln 2}{a}}$) is dependent on propagation distance z! (when $\beta \neq 0$)

Simpliest example: $b_0 = 0$ (no initial chirp on pulse, i.e. transform-limited)

$$a(z) = \frac{a_0}{1 + 4a_0^2\beta''^2 z^2}$$

or

$$\tau_p = \sqrt{\frac{2\ln 2}{a_0}(1 + 4a_0^2\beta''^2 z^2)}$$

$$= \tau_{p0}\sqrt{1 + 4a_0^2\beta''^2 z^2}$$

$$(\tau_p = \sqrt{2}\tau_{p0} \text{ in distance } l = \frac{1}{2a_0\beta''})$$

clearly, the pulse width **increases** with z.

The rate of increase is determined by

(1) $\beta''(\omega_0)$. Some authors call β'' the **group velocity dispersion** (GVD). Others define the GVD according to

$$GVD = \frac{dv_g}{d\omega} = \frac{d}{d\omega}[\frac{1}{(\frac{d\beta}{d\omega})}]$$

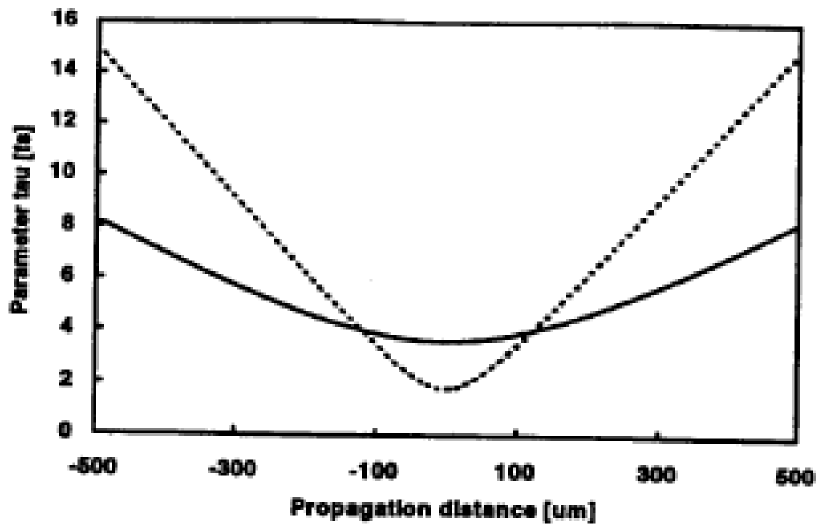

Figure 4: $\tau(z)$ for a gaussian pulse of respectively minimal duration of 6 fs (continuous line) and 3 fs (dashed line), for propagation in fused silica.

$$= \frac{-1}{(\frac{d\beta}{d\omega})^2} \frac{d^2\beta}{d\omega^2} = -v_g^2 \frac{d^2\beta}{d\omega^2}|_{\omega_0} = GVD(\omega_0)$$

In either case, $\beta'' \neq 0 \Rightarrow$ pulse width changes with propagation distance.

(2) a_0. Recall pulsewidth $\tau_{p0} \sim \frac{1}{\sqrt{a_0}} \leftrightarrow$ *bandwidth* $\Delta\omega_p \sim \sqrt{a_0}$. That is, the pulse broadening is larger for a shorter pulsewidth, as one would expect.

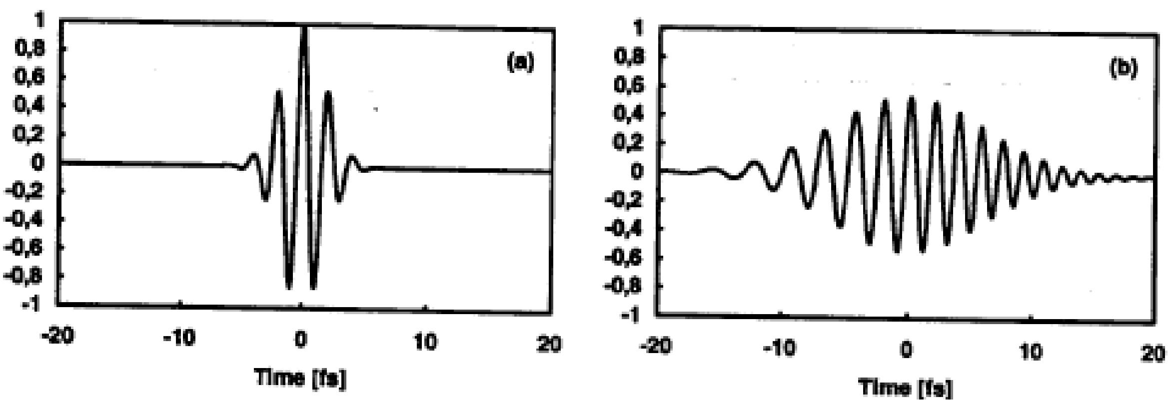

Figure 3: Electrical field of a 3 fs pulse (a), and after propagation in 200 μm of fused silica (b). In this last case the duration has become 10 fs.

Physical interpretation of GVD:

Recall that v_g is the **slope** $\frac{d\omega}{d\beta}$(or $\frac{d\omega}{dk}$), and that there is a 'phase slip' between the phase fronts and the envelope. However, the pulse enveloep stays together if $\frac{d\omega}{d\beta}$ is constant.

If $\frac{d^2\omega}{d\beta^2} \neq 0$, then we must consider the pulse to be made up of a series of **wavepackets**, each with a different ω_0. Each wavepacket will have its own $\frac{d\omega}{d\beta}$(goes at its own v_g), so the pulse envelope changes with propagation.

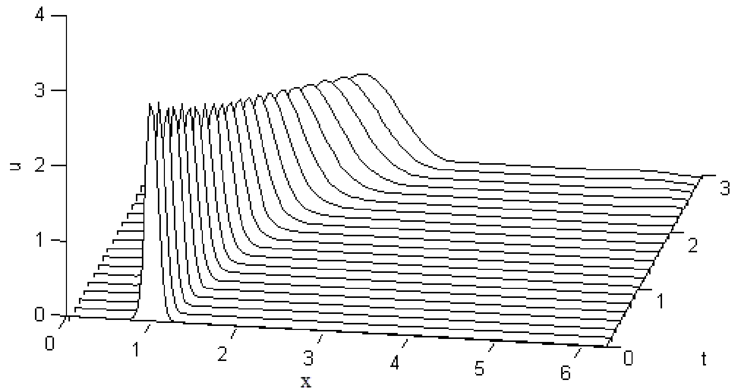

Figure 5: Pulse shape of propagation of a linearly chirped Gaussian pulse in a GVD medium.

To think about: why does not linear variation of β cause pulse spreading?

Now go back to the time domain and consider the pulse to be composed of different wavepackets (which are longer in duration!!) and centered at different frequencies ($\omega_{01}, \omega_{02}, \omega_{03}$, *etc.*) (Figure 6)

0.1 Simple model of GVD:

Consider a pulse (of any smooth shape) of spectral width $\Delta\omega_p$ and duration τ_p. Each spectral component travels with its own v_g, and the difference of goup velocities over the spectrum is

$$\Delta v_g \simeq \frac{dv_g}{d\omega}|_{\omega_0} \Delta\omega_p$$

Thus after propagating a distance z, the pulse will spread by an amount

$$\Delta\tau_p = \Delta(\frac{z}{v_g}) = -\frac{z}{v_g^2}\Delta v_g$$

$$= -\frac{z}{v_g^2}\frac{dv_g}{d\omega}\Delta\omega_p$$

$$= -\frac{z}{v_g^2}(-v_g^2\frac{d^2\beta}{d\omega^2})\Delta\omega_p$$

$$= z\frac{d^2\beta}{d\omega^2}\Delta\omega_p$$

Frequently, this is written in the form

$$\Delta\tau_p = -DL\Delta\lambda$$

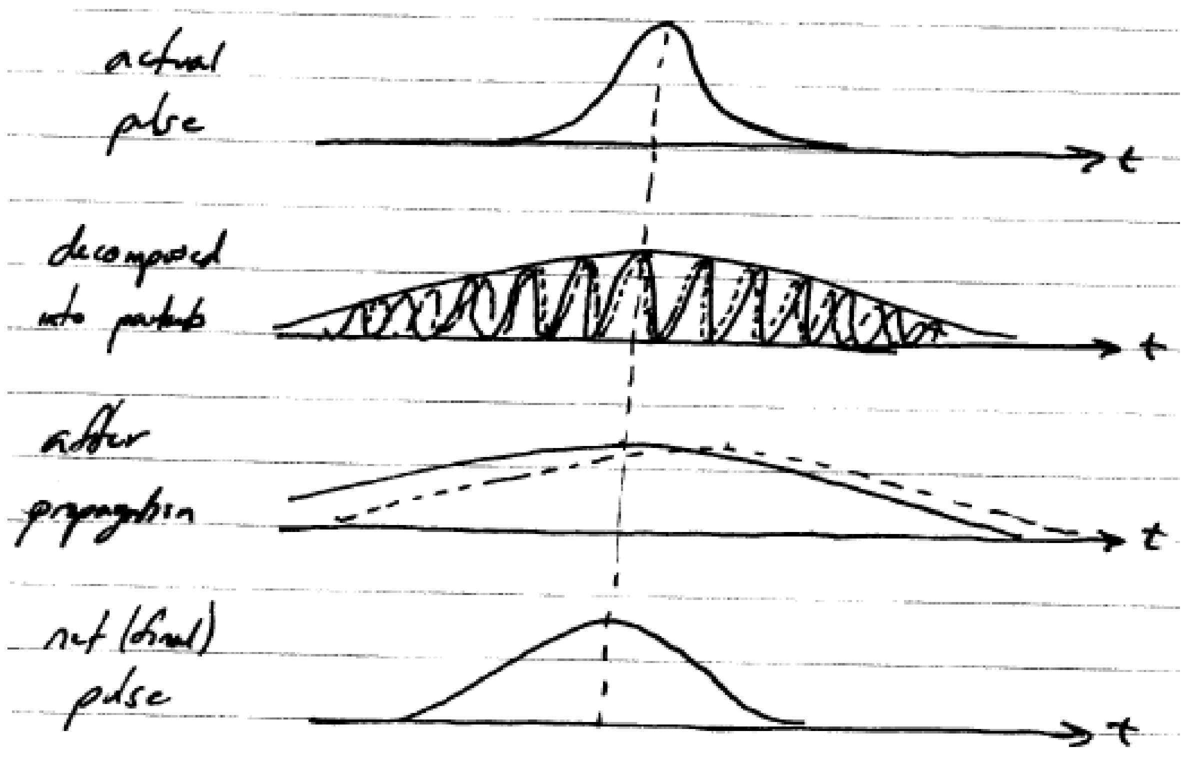

Figure 6: Wavepacket decomposition of pulses.

D is in $\frac{ps}{m \cdot nm}$. Since $\Delta\omega_p = -\omega_0 \frac{\Delta\lambda}{\lambda}$, $\Rightarrow D = -\frac{d^2\beta}{d\omega^2}\frac{\omega}{\lambda}$