### 0.1 Positive dispersion with gratings

There are two typical situations in which we would like to use gratings to obtain positive dispersion:

1. (the reason they were invented by O. Martinez)

In the important wavelength region of $1.5 \mu \mathrm{~m}$ for optical fiber, the dispersion is actually negative (it flips sign at about $1.3 \mu m$ ), i.e. $\frac{d^{2} n}{d \lambda^{2}}<0$ for $\lambda>1.3 \mu m$ in silica glass fiber.

$$
\Rightarrow \beta^{\prime \prime}{ }^{i} 0 \Longrightarrow D=-\frac{\omega}{\lambda} \beta^{\prime \prime}>0 \text { (Figure 1) }
$$

(The negative dispersion is a consequence of infrared absorption in the glass: see Rgure?? 22.1-2 from Saleh+Teich)


Figure 1: Negative dispersion for $\lambda>1.3 \mu m$.

Thus, in order to compensate material dispersion that is negative, we need something with adjustable positive dispersion.
2. Controllable pulse chirping(especially for 'chirped pulse amplification').

We would like to be able to stretch and compress pules in a controllable way, without using any material (so that nonlinear effects due to the high peak power of ultrashort pulses can be avoided).

In fact, there is a scheme that can give controllable disperison of either sign using gratings. Note that $\varphi^{\prime \prime}$ for our grating pair is always negative for grating separation $G>0$. A positive $\varphi^{\prime \prime}$ could therefore be obtained. if we could make $G<0$.

A negative effeective separation can be obtained by using images. The basic idea is to make an image of the first grating that is behind the second grating; that will give an effective $G<0$.

Simple idea to make $G=0$ : image grating 1 onto grating 2.(Figure 2) Clearly, the optical path length (and hence the group delay) is the same for all wavelengths $\Rightarrow$ no dispersion.


Figure 2: Simple idea to make $G=0$ : image grating 1 onto grating 2.

$$
G=0 \Rightarrow \varphi^{\prime \prime}=0
$$

(At first glance, this looks like an expensive way to do nothing, but we will see later that this setup is actually quite useful!)

Now consider what happens as the gratings are moved in towards the lences.(Figure 3) The effective propagation distance is now $b=z_{1}+z_{2}$ (and $\left.b=\frac{G}{\cos \theta}\right)$.


Figure 3: the gratings are moved in towards the lences.
$\Rightarrow G<0$ if gratings are closer to lenses than $f .(\Rightarrow$ image of 1 is behind 2$)$
$G>0$ if gratings are farther from lenses than $f$.
All the dispersion eqns. we have derived apply to this grating scheme, but now with an effective separation $G$ which can be positive, negative, or zero.

### 0.2 Angular Dispersion in general

## Transverse Group Delay

Let us consider first the tilting of a pulse front that occurs on refraction. We need to recall from last time that

$$
\begin{gathered}
\text { phase delay } t_{p}=\frac{\varphi\left(\omega_{0}\right)}{\omega_{0}} \\
\varphi^{\prime}\left(\omega_{0}\right)=t_{g}=\text { group delay }=\frac{z}{c}\left(n-\lambda \frac{d n}{d \lambda}\right)
\end{gathered}
$$

Note that in normal optical materials $\frac{d n}{d \lambda}<0 \Longrightarrow$ group delay is larger than the phase delay $\Longrightarrow$ the pulse envelope lags behind the wavefront advance.

Now consider a beam of finite size incident on glass, where the pulse and wavefronts are initially lined up.(Figure 4) In the time $B \rightarrow B^{\prime}, A \rightarrow A^{\prime}$. That elapsed time is

$$
T_{\text {phase }}=\frac{n}{c} \overline{A A^{\prime}}=\frac{\overline{B B^{\prime}}}{c}=\frac{D \tan \tau}{c} \text { from diagram }
$$



Figure 4: beam of finite size incident on glass.

Note: pulse and phase fronts coincide at $B^{\prime}$ as well as at $B$ (propagation in nodispersive medium).

But: glass is dispersive, and in that time the phase front has advanced to $A^{\prime}$, but the pulse has advanced only to $E$ (longer group delay, or slower group velocity compared to phase velocity). $\Longrightarrow$ a tile angle $\sigma$ builds up. $\tan \sigma=\frac{\overline{E A^{\prime}}}{D^{\prime}}$.

Generally:

$$
\begin{gathered}
\overline{E A^{\prime}}=\left(\frac{c}{n}-v_{g}\right) T_{\text {phase }} \\
=\left(\frac{c}{n}-v_{g}\right) \frac{D \tan \tau}{c}
\end{gathered}
$$

also

$$
\frac{D^{\prime}}{\cos \gamma^{\prime}}=\frac{D}{\cos \gamma} \Longrightarrow D^{\prime}=\frac{D \cos \gamma^{\prime}}{\cos \gamma}=\frac{D \sqrt{1-\sin ^{2} \gamma^{\prime}}}{\cos \gamma}
$$

Snell: $\sin \gamma=n \sin \gamma^{\prime} \Longrightarrow D^{\prime}=\frac{D \sqrt{n^{2}-\sin ^{2} \gamma}}{n \cos \gamma}$

$$
\begin{gathered}
\Longrightarrow \tan \sigma=\frac{\overline{E A^{\prime}}}{D^{\prime}}=\frac{\left(\frac{c}{n}-v_{g}\right) \frac{D \tan \tau}{c}}{\frac{D \sqrt{n^{2}-\sin ^{2} \gamma}}{n \cos \gamma}} \\
\quad=\left(\frac{c}{n}-v_{g}\right) \frac{n}{c} \frac{\sin \gamma}{\sqrt{n^{2}-\sin ^{2} \gamma}}
\end{gathered}
$$

This is probably the most useful form, since it makes the physical meaning clear. The difference between the phase and group velocities results in a tilt, or transverse group delay upon refraction into a dispersive medium. The first part of the expression shows that the tilt angle gets larger for more highly dispersive media. The second part of the expression just says the angle $\sigma$ bigger at larger incidence angles where the beam deviation is larger.

## Propagation through plane-parallel plate:

- pulse front acquires tilt inside plate, but on exiting the 2nd refraction corrects the tilt.
- augument: just apply time reversal.
- ?? the pulses amplitude and phase fronts are again parallel.
- if the planes are not parallel(i.e. a prism), the pulse will acquire a tilt!

Note that the tile angle depends on the group velocity. Thus if a broadband pulse is incident on a prism, diffirent frequency components will have different $v_{g}$ due to GVD $\Rightarrow$ different $\sigma$.

Consider the pulse front at the output of a prism.(Figure 5)


Figure 5: pulse front at the output of a prism.
$\overline{E A^{\prime}}=$ distance phasse has traveled compared to the pulse front $=\left(\frac{a}{v_{p}}-\frac{a}{v_{g}}\right) c$.
$\left(\frac{a}{v_{p}}-\frac{a}{v_{g}}\right)$ is difference in travel time across the prism.

$$
\begin{gathered}
\frac{1}{v_{p}}=\frac{n}{c}, \frac{1}{v_{g}}=\frac{n}{c}-\frac{\lambda}{c} \frac{d n}{d \lambda} \\
\overline{E A^{\prime}}=a c\left(\frac{n}{c}-\frac{n}{c}+\frac{\lambda}{c} \frac{d n}{d \lambda}\right)=a \lambda \frac{d n}{d \lambda} \\
\tan \sigma=\frac{a \lambda}{D^{\prime}} \frac{d n}{d \lambda}
\end{gathered}
$$

This is in terms of the beam + prism sizes; what we expect is that it should depend only on the angular deviation $\theta$. We just need to recall some simple prism geometry. At minimum deviation, the angle of deviation $\theta$ is given by (see, e.g. Hecht's Optics)

$$
n \sin \frac{\gamma}{2}=\sin \frac{\theta+\gamma}{2}(\gamma=\text { apex angle })
$$

also

$$
\cos \frac{\theta+\gamma}{2}=\frac{D^{\prime}}{L}
$$

(at minimum deviation, the ray is parallel to the prism base)

$$
L \sin \frac{\gamma}{2}=\frac{a}{2}
$$

taking derivative

$$
\begin{gathered}
\frac{d n}{d \lambda} \sin \frac{\gamma}{2}=\frac{1}{2} \cos \frac{\theta+\gamma}{2} \frac{d \theta}{d \lambda} \\
\frac{d \theta}{d \lambda}=\frac{2 \sin \frac{\gamma}{2}}{\cos \frac{\theta+\gamma}{2}} \frac{d n}{d \lambda}=\frac{a}{L} \cdot \frac{L}{D^{\prime}} \cdot \frac{d n}{d \lambda} \\
\frac{d \theta}{d \lambda}=\frac{a}{D^{\prime}} \cdot \frac{d n}{d \lambda} \text { (angular dispersion) }
\end{gathered}
$$

$\Rightarrow$ we can write the tilt angle as

$$
\tan \sigma=\lambda \frac{d n}{d \lambda}
$$

Bor has shown that in fact this is a general relation: an angular dispersion $\frac{d \theta}{d \lambda}$ lends to a transverse group delay.

### 0.3 GVD through angular dispersion

Recall the general program for pulse propagation: for a lossless medium we just need the phase part of the transfer function $e^{-i \varphi(\omega)}$, where the phase delay is

$$
\varphi(\omega)=\beta(\omega) z=\frac{\omega}{c} n(\omega) z=\frac{\omega}{c} p(\omega)
$$

where $p(\omega)$ is the optical path length.
Thus the group delay dispersion is

$$
\varphi^{\prime \prime}\left(\omega_{0}\right)=\left.\frac{d^{2} \varphi}{d \omega^{2}}\right|_{\omega_{0}}=\left.\left\{\frac{\omega}{c} \frac{d^{2} p}{d \omega^{2}}+\frac{2}{c} \frac{d p}{d \omega}\right\}\right|_{\omega_{0}}
$$

$$
=\left.\frac{\lambda^{3}}{2 \pi c^{2}} \frac{d^{2} p}{d \lambda^{2}}\right|_{\lambda_{0}}
$$

Let us consider propagation of a pulse through some arbitrary optical element which causes angular dispersion, which we can assume to occur at point $O$. (Figure 6) The pulse's central frequency $\omega_{0}$ goes in thee direction $\vec{r}\left(\omega_{0}\right)$, with a phase front $S_{0}^{\prime}$. Some other frequency component as within the pulse bandwidth goes at an angle $\alpha$ with respect to $\omega_{0}$, with a phase front $S^{\prime}$. Of course, both frequency components have parallel phase fronts $S_{0}, S$ on the left side.


Figure 6: propagation of a pulse through some arbitrary optical element which causes angular dispersion at point $O$.

Consider $P_{0}=$ point of reference $\Rightarrow$ wavefront $S^{\prime}$ intersects $S$ at $P_{0}$.
Optical path length $\overline{O P^{\prime}}=p(\omega)=p\left(\omega_{0}\right) \cos \alpha \equiv L \cos \alpha \Rightarrow \frac{\omega L}{c} \cos \alpha(\omega)$

$$
\begin{gathered}
\frac{d \varphi}{d \omega}=\frac{L}{c} \cos \alpha-\frac{\omega L}{c} \sin \alpha \frac{d \alpha}{d \omega} \\
\left.\frac{d^{2} \varphi}{d \omega^{2}}\right|_{\omega_{0}}=-\frac{L}{c} \sin \alpha \frac{d \alpha}{d \omega}-\frac{L}{c} \sin \alpha \frac{d \alpha}{d \omega}-\frac{\omega L}{c} \cos \alpha\left(\frac{d \alpha}{d \omega}\right)^{2}-\frac{\omega L}{c} \sin \alpha \frac{d^{2} \alpha}{d \omega^{2}} \\
=-\left.\frac{L}{c}\left\{\sin \alpha\left[2 \frac{d \alpha}{d \omega}+\omega \frac{d^{2} \alpha}{d \omega^{2}}\right]+\omega \cos \alpha\left(\frac{d \alpha}{d \omega}\right)^{2}\right\}\right|_{\omega_{0}}
\end{gathered}
$$

This is the group delay dispersion evaluated at the carrier frequency $\omega_{0}$. Thus
$\sin \alpha(\omega)$ and $\cos \alpha(\omega)$ are evaluated at $\omega_{0}$, so we have $\alpha\left(\omega_{0}\right)=0 \Rightarrow \sin \alpha=0, \cos \alpha=1$ exactly. (Note that the treatment in Diels + Rudalph 2.6.2 is not quite correct, since they simply make an approximation $\sin \alpha \simeq 0$ and $\cos \alpha \simeq 1$ corresponding to small $\alpha$ over the pulse bandwidth.)

$$
\left.\frac{d^{2} \varphi}{d \omega^{2}}\right|_{\omega_{0}}=-\frac{L \omega_{0}}{c}\left(\left.\frac{d \alpha}{d \omega}\right|_{\omega_{0}}\right)^{2} \text { (exactly) }
$$

$\Rightarrow$ angular dispersion results in GDD.
(note that, the GDD is always negative, since the dispersion comes in as square) note also $\mathrm{GDD} \propto L$, i.e. the distance from the dispersing optical element.

From the above, we can also obtain the third order phase term:

$$
\begin{gathered}
\left.\frac{d^{3} \varphi}{d \omega^{3}}\right|_{\omega_{0}}=-\left.\frac{L}{c}\left\{\cos \alpha\left[3\left(\frac{d \alpha}{d \omega}\right)^{2}+3 \omega \frac{d \alpha}{d \omega} \frac{d^{2} \alpha}{d \omega^{2}}\right]+\sin \alpha\left[3 \frac{d^{2} \alpha}{d \omega^{2}}+\omega \frac{d^{3} \alpha}{d \omega^{3}}-\omega\left(\frac{d \alpha}{d \omega}\right)^{3}\right]\right\}\right|_{\omega_{0}} \\
=-\left.\frac{3 L}{c}\left[\left(\frac{d \alpha}{d \omega}\right)^{2}+\omega \frac{d \alpha}{d \omega} \frac{d^{2} \alpha}{d \omega^{2}}\right]\right|_{\omega_{0}}
\end{gathered}
$$

Therefore, we have the required phase terms once we know $\alpha(\omega)$ foor the specific optical element desired.

Of course (as we saw with gratings), we do not want to have a beam divergence associated with the GDD, so we will usually use pairs of dispersive elements.

