Lecture 35

Thus we have ,from (*) Ｐ．３５７，

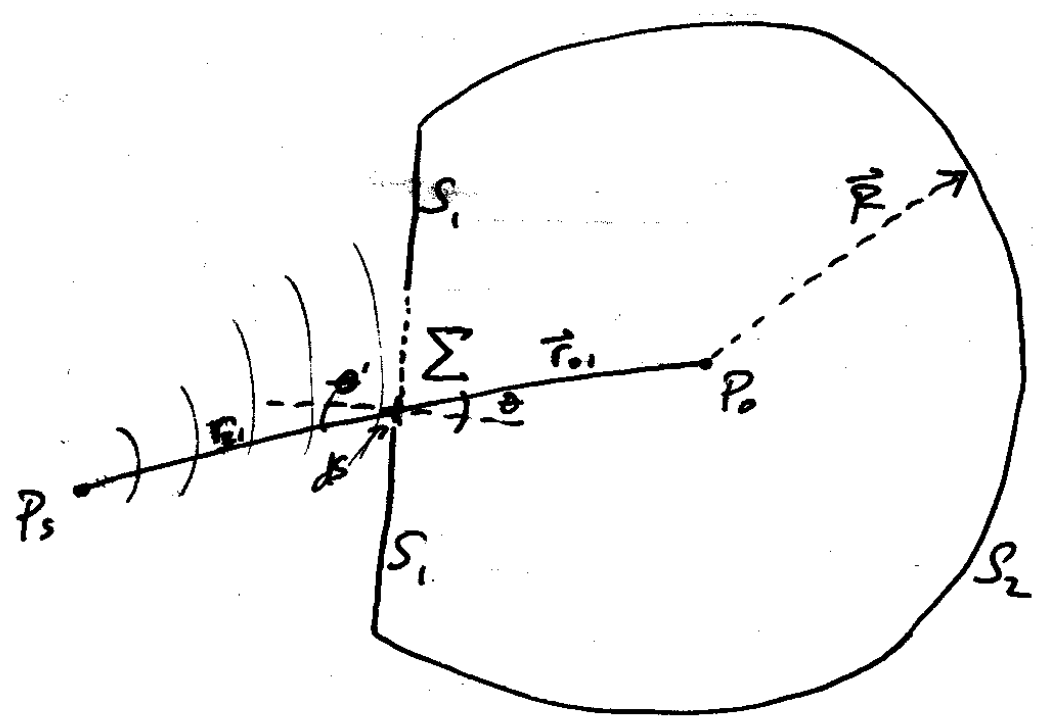
$$\varphi(p_0)=\frac{1}{4\pi}\iint_S\left[\frac{e^{-i\vec{k}\cdot\vec{r_{01}}}}{r_{01}}\frac{\partial\varphi}{\partial n}-\varphi\frac{\partial}{\partial n}\left(\frac{e^{-i\vec{k}\cdot\vec{r_{01}}}}{r_{01}}\right)\right]ds$$

This is the Helmholtz-Kirchhoff integral theorem.

It says we know the field at P_0 given its value (and thus its derivative $\frac{\partial\varphi}{\partial n}$ also) on any surface S containing P_0.

Now suppose we have a point source at P_s, and we want the field at P_0, but there is an opaque screen containing an aperture between them:

We take our surface S enclosing P_0 to be as shown:



Thus we need

$$\iint_S=\iint_{S_1}+\iint_{S_2}+\iint_\Sigma$$

In order to proceed any further we must now make some approximations.

1. Assume S_2 can be made far enough from P_0 that φ and $\frac{\partial\varphi}{\partial \text{n}}$ are zero on S_2.When is this assumption valid?:

   On S_2, $\frac{e^{-ik\cdot R}}{R}$

$$\frac{\partial \varphi}{\partial \mathrm{n}} = (-ik - \frac{1}{R})\frac{e^{-ik\cdot R}}{R} \simeq -ik\frac{e^{-ik\cdot R}}{R} \quad \text{as R } \to \infty$$

$$\iint_{S_2}\left[\frac{e^{-ik\cdot R}}{R}\frac{\partial \varphi}{\partial \mathrm{n}} - \varphi\left(-ik\frac{e^{-ik\cdot R}}{R}\right)\right]dS = \iint_{S_2} R\left[\frac{\partial \varphi}{\partial \mathrm{n}} + ik\varphi\right]e^{-ikR}d\Omega, dS = R^2 d\Omega$$

$\to 0$ as $R \to \infty$ iff

$$\lim_{R\to\infty} R\left[\frac{\partial \varphi}{\partial \mathrm{n}} + ik\varphi\right] =$$

This is called the Sommerfield radiation condition.

It is straightforward to show that this condition is satisfied by a spherical wave. If we consider the field over an arbitrary aperture to be a superposition of spherical waves, then this condition is generally satisfied.

2. Assume that the field φ and its derivative $\frac{\partial \varphi}{\partial \mathrm{n}}$ in Σ are identical to what they would be without the screen S_1.

3. Assume that $\varphi = 0$ on the screen S_1. (i.e. the screen is perfectly opaque)

   The latter two assumptions are known as Kirchhoff's boundary conditions.

   The boundary conditions are certainly reasonable for all regions of the screen S_1 and in the aperture Σ, except near the boundary between S_1 and Σ !

   It is not obvious that these assumptions are valid near the boundary , since (i) the boundary conditions on the field are clearly violated, and (ii) the total field is the sum of the field incident on the aperture plus the field generated by the polarization induced on the screen boundary by the incident field —indeed, it is the sum of these two fields that produces the diffracted field in rigorous diffraction theory.

In fact, Kirchhoff's boundary conditions are not valid for a region within a few wavelengths of the boundary.
More rigorous treatments of diffraction show that the Kirchhoff boundary conditions yield accurate solutions provided

(i) The size of the aperture $\kappa \gg \lambda$

(ii) The observation point P_0 is far ($\gg \lambda$) from the aperture.

If these conditions are not satisfied, a full boundary-value problem must be solved. From now on, we will assume such "fringing field" effects can be safely neglected.
With the Kirchhoff boundary conditions, the Helmholtz-Kirchhoff integral formula reduces to

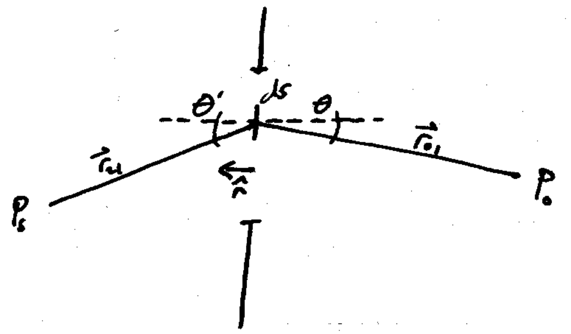
$$\varphi(P_0)=\frac{1}{4\pi}\iint_\Sigma\left[\frac{e^{-i\vec{k}\cdot\overrightarrow{r_{01}}}}{r_{01}}\frac{\partial\varphi}{\partial n}-\varphi\frac{\partial}{\partial n}\left(\frac{e^{-i\vec{k}\cdot\overrightarrow{r_{01}}}}{r_{01}}\right)\right]ds$$

Now if we have a point source at P_S, and if the aperture Σ is far from P_S, then in the aperture

$$\varphi=\frac{e^{-i\vec{k}\cdot\overrightarrow{r_{21}}}}{r_{21}}$$ (i.e. source field $E=E_0\varphi e^{-i\omega t}$)

$$\frac{\partial\varphi}{\partial \mathrm{n}}=\left(-i\vec{k}\cdot\hat{n}-\frac{1}{r_{21}}\right)\frac{e^{-i\vec{k}\cdot\overrightarrow{r_{21}}}}{r_{21}}\simeq-i\vec{k}\cdot\hat{n}\frac{e^{-i\vec{k}\cdot\overrightarrow{r_{21}}}}{r_{21}}\left(r_{21}\gg\frac{1}{k}\right)$$

Looking in the aperture region a little more closely:



⇨ On left hand side $\vec{k}\cdot\hat{n}=-k\cos\theta'$ (minus since $\vec{k}$ points to the right)

$$-i\vec{k}\cdot\hat{n}=ik\cos\theta'$$

Similarly,

$$\frac{\partial}{\partial n}\left(\frac{e^{-i\vec{k}\cdot\overrightarrow{r_{01}}}}{r_{01}}\right)\simeq+ik\cos\theta$$

Thus $$\varphi(P_0)\simeq\frac{E_0}{4\pi}\iint_\Sigma\left(ik\cos\theta'+ik\cos\theta\right)\frac{e^{-i\vec{k}\cdot(\overrightarrow{r_{01}}+\overrightarrow{r_{21}})}}{r_{01}r_{21}}ds$$

(*) $$\boxed{\varphi(P_0)\simeq+\frac{i}{\lambda}E_0\iint_\Sigma\left(\frac{\cos\theta'+\cos\theta}{2}\right)\frac{e^{-i\vec{k}\cdot(\overrightarrow{r_{01}}+\overrightarrow{r_{21}})}}{r_{01}r_{21}}ds}$$

= <u>Fresnel – Kirchhoff Diffraction Formula</u>

It is useful to rewrite this in the following form:

$$\varphi(P_0)=\iint_\Sigma F(\theta,\theta')\frac{e^{-i\vec{k}\cdot\overrightarrow{r_{01}}}}{r_{01}}ds$$

Where

$$F(\theta,\theta')=+\frac{i}{\lambda}E_0\frac{e^{-i\vec{k}\cdot\overrightarrow{r_{21}}}}{r_{21}}\left(\frac{\cos\theta'+\cos\theta}{2}\right)$$

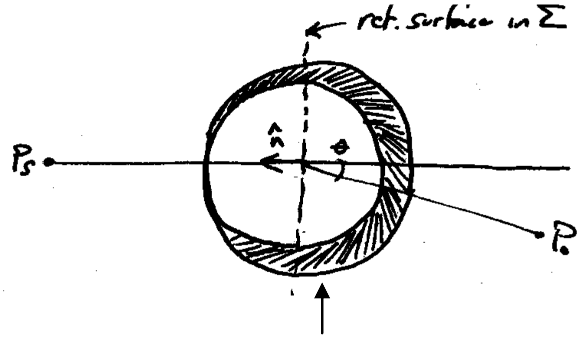
There are several features to note:

1. F looks like a "source term" in the aperture, and the integral over Σ means the field at P_0 is the integral over all the "fictitious point sources "located Σ.

2. The strength of the secondary sources is E_0 / λ (field per unit wavelength).

3. There is a $90°$ phase shift between the radiated field and the secondary sources. This is physically just the Gouy phase shift that any field undergoes in propagating from the near field to the far field.

4. The term in brackets $\left(\frac{\cos\theta' + \cos\theta}{2}\right)$ is called the obliquity factor. It says that the fictitious secondary sources are not in fact spherical radiators , but they radiate preferentially in the forward direction , as seen in the following plot :
   Consider source on axis:



Thickness of shaded region signifies source strength

Note $\frac{\cos\theta' + \cos\theta}{2} = \frac{1+1}{2} = 1$ in forward direction

$\frac{\cos\theta' + \cos\theta}{2} = \frac{-1+1}{2} = 0$ in backward direction

Thus the obliquity factor solves one serious problem with the "intuitive" formulation of Huygen's Principle, namely the presence of the "backwards wave."
Huygen's wavelets which are spherical waves generate both forward and backward propagating waves. Our integral treatment shows that the backward wave actually does not exist (the obliquity factor eliminates it).

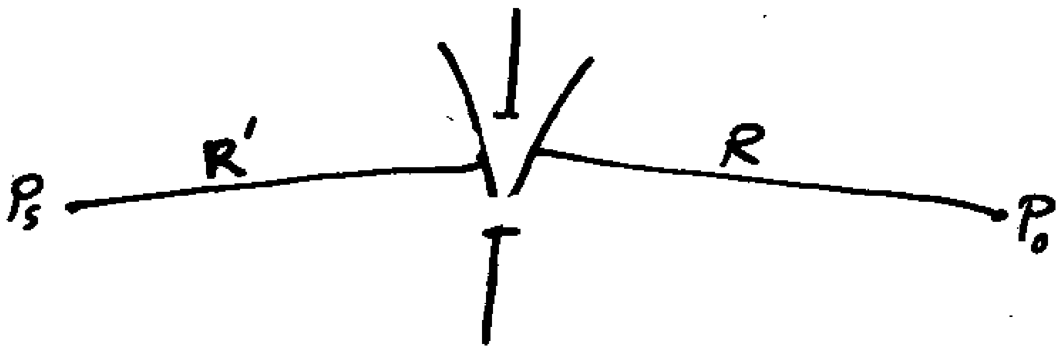
Diffraction of Paraxial Waves

The principle assumptions underlying the Fresmal-Kirchhoff theory are that

aperture – observation distance $R \gg \lambda$

aperture dimensions $d > \lambda$.

We will now restrict our attention to diffraction of paraxial waves, where the wavefronts travel essentially along the z-axis and are spherical.



Paraxial approximation: $d \ll R$ and $d \ll R'$
(aperture "looks small" from <u>either</u> the source or the observation point)

- In this case , r_{01} and r_{21} are essentially R and R' respectively (they vary by only a small amount over the aperture) and thus can be taken out of the Fresnel-Kirchhoff formula in the denominator
- Likewise the obliquity factor is essentially constant over the aperture

The Fresnel-Kirchhoff formula for paraxial rays is thus

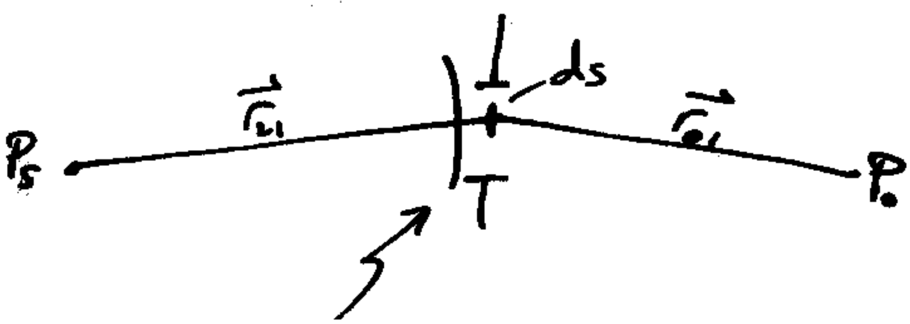
$$\varphi(P_0) = +i\frac{E_0}{\lambda}\left(\frac{\cos\theta' + \cos\theta}{2}\right)\frac{1}{RR'}\iint_\Sigma e^{-i\vec{k}\cdot(\overrightarrow{r_0}+\overrightarrow{r})}ds$$

Note that we <u>cannot</u> take the $\overrightarrow{r_{01}} + \overrightarrow{r_{02}}$ in the phase term outside the integral since the phase can vary by many π over the aperture.
In all over calculations, we will also take the obliquity factor to be essentially unity. (It turns out that the error this induces is less than 5% for $\theta, \theta' < 18°$. Ref: Goodman).

$$\frac{\cos\theta' + \cos\theta}{2} \simeq \frac{1+1}{2} = 1$$

$$\varphi(p_0) \simeq +i\frac{E_0}{\lambda RR'}\iint_\Sigma e^{-i\vec{k}\cdot(\overrightarrow{r_{01}}+\overrightarrow{r_{21}})}ds$$

This describes the diffraction of a paraxial wave emitted at a point source P_S as observed at P_0.



Spherical wave at input described by $\dfrac{e^{-i\vec{k}\cdot\overrightarrow{r_{21}}}}{R'}$

Quite commonly, of course, we must consider input waves which are more complicated. We may consider an arbitrary input field at the input to the aperture as a simple extension of the single –point-source field. There are many different notations in the various texts, but we will follow one fairly close to Guenthers.
We suppose the field immediately after the aperture is given by

$$E_0 f\left(x, y\right)$$

Where $f\left(x, y\right)$ is the complex aperture function, describing the (possibly complicated) variation in amplitude and phase over the aperture. The diffraction integral is now

(*) $$\boxed{E\left(P_0\right) = +i\frac{E_0}{\lambda R}\iint_\Sigma f\left(x, y\right)e^{-i\vec{k}\cdot\overrightarrow{r_{01}}}ds}$$

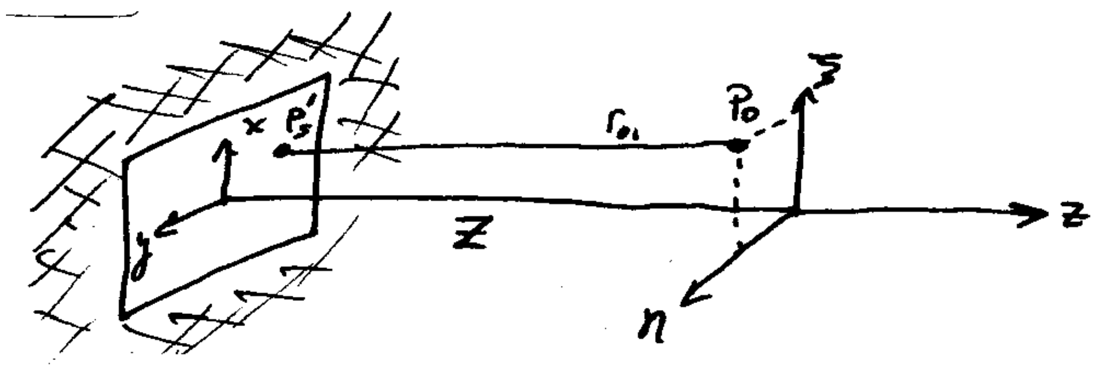
This is the integral we must solve to find the field at P_0 given $E_0 f\left(x, y\right)$.

Geometry: $\left(x, y\right)$ = coordinates in aperture

$\left(\xi, \eta\right)$ =coordinates in observation plane

R= length from elements ds to p_0

Z= distance between $\left(x, y\right)$ and $\left(\xi, \eta\right)$ planes



$$r_{01} = \sqrt{\left(x-\xi\right)^2 + \left(y-\eta\right)^2 + z^2}$$

In the paraxial approximation

$$r_{01} = z\sqrt{\left(\frac{x-\xi}{z}\right)^2 + \left(\frac{y-\eta}{z}\right)^2 + 1} \simeq z + \frac{\left(x-\xi\right)^2 + \left(y-\eta\right)^2}{2z}$$

Keeping only the first term in the binominal expression
This is known as the Fresnel approximation.
We can easily see that this is the same Fresnel approximation as we made in our consideration of the paraxial wave eqn. (see notes P.331)

- Note $r_{01} \simeq z$ ok for wave amplitude ,as usual

- Plug above in (*) P.367 =>