0.1 Pulse propagation from a more general standpoint:

Parabolic equation

We know that the wave eqn. describes the propagation of the field in a dielectric medium

$$(\frac{\partial^2}{\partial z^2} - \mu_0\epsilon_0\frac{\partial^2}{\partial t^2})\tilde{\varepsilon}(z,t) = \mu_0\frac{\partial^2\tilde{p}}{\partial t^2}$$

where $\tilde{p}(z,\omega) = \epsilon_0\tilde{\chi}(\omega)\tilde{\varepsilon}(z,\omega)$ is the linear polarization in the medium. We know, however, that we can find solutions that look like

$$\tilde{\varepsilon}(z,t) = Re\{\tilde{E}(z,t)e^{i[\omega_0 t-\beta(\omega_0)z]}\}$$

$\tilde{E}(z,t)$: complex envelope; $e^{i[\omega_0 t-\beta(\omega_0)z]}$: carrier wave.

Thus it should be very useful to obtain a wave equation which describes the propagation of the (complex) envelope. We should be able to do this since we know the function $\beta(\omega)$ (remember that this comes from the expression for $\tilde{\chi}(\omega)$].

The approach taken by Siegman in section 7.2 is to expand $\tilde{\chi}(\omega)$ in the vacinity of ω_0, and make extensive substitution back into the wave eqn. We will take a more heevrishe?? approach (due to Haus), but will arrive at the same result.

We know that one frequency component of the field propagate as

$$\tilde{\varepsilon}(z,\omega) = \tilde{\varepsilon}(0,\omega)e^{-i\beta(\omega)z}$$

so the field obeys the differential equation

$$\frac{\partial\tilde{\varepsilon}}{\partial z} = -i\beta\tilde{\varepsilon}$$

We will assume the pulse to have a sufficiently narrow spectrum that a Taylor expansion of $\beta(\omega)$ can be performed:

$$\frac{\partial \tilde{\varepsilon}}{\partial z} = -i\{\beta(\omega_0) + \frac{d\beta}{d\omega}|_{\omega_0}(\omega - \omega_0) + \frac{1}{2}\frac{d^2\beta}{d\omega^2}|_{\omega_0}(\omega - \omega_0)^2\}\tilde{\varepsilon} \tag{1}$$

Now, if the spectrum of $\tilde{\varepsilon}$ is narrow and centered at ω_0, it is convenient to express it as a function of $(\omega - \omega_0)$

$$\tilde{\varepsilon}(z, \omega) = \tilde{E}(z, \omega - \omega_0)e^{-i\beta(\omega_0)z} \tag{2}$$

where $\tilde{E}$ is the complex envelope (complex since there may be phase shifts relative to the carrier $\beta(\omega_0)$). Note that $\tilde{E}$ is assumed to vary slowly *w.r.t.* z.

Time domain:

$$\tilde{\varepsilon}(z, t) = \frac{1}{2\pi}\int_{-\infty}^{+\infty} \tilde{\varepsilon}(z, \omega)e^{i\omega t}d\omega$$

$$= \frac{1}{2\pi}\int_{-\infty}^{+\infty} \tilde{E}(z, \omega - \omega_0)e^{-i\beta(\omega_0)z}e^{i\omega t}d\omega$$

$$= \frac{1}{2\pi}e^{i[\omega_0 t - \beta(\omega_0)z]}\int_{-\infty}^{+\infty} \tilde{E}(z, \omega - \omega_0)e^{i(\omega - \omega_0)t}d\omega$$

$$= e^{i[\omega_0 t - \beta(\omega_0)z]}\tilde{E}(z, t) \tag{3}$$

i.e. $\tilde{E}(z, t)$ = slowly varying complex envelope ?? in time domain= Fourier transform of $\tilde{E}(z, \omega - \omega_0)$.

Next step: convert the differential eqn. for $\tilde{\varepsilon}(z, \omega)$ into one for $\tilde{E}(z, \omega)$.

Substituting Eqn. (2) into Eqn. (1),

$$\frac{\partial}{\partial z}[\tilde{E}(z, \omega - \omega_0)e^{-i\beta(\omega_0)z}] = -i\{\beta(\omega_0) + \frac{d\beta}{d\omega}|_{\omega_0}(\omega - \omega_0) + \frac{1}{2}\frac{d^2\beta}{d\omega^2}|_{\omega_0}(\omega - \omega_0)^2\}\tilde{E}(z, \omega - \omega_0)e^{-i\beta(\omega_0)z}$$

$$[\frac{\partial}{\partial z}\tilde{E} - i\beta(\omega_0)\tilde{E}]e^{-i\beta(\omega_0)z} = -i\{\beta(\omega_0) + \frac{d\beta}{d\omega}|_{\omega_0}(\omega-\omega_0) + \frac{1}{2}\frac{d^2\beta}{d\omega^2}|_{\omega_0}(\omega-\omega_0)^2\}\tilde{E}e^{-i\beta(\omega_0)z}$$

$$divide\, by\, e^{-i\beta(\omega_0)z}\, and\, cancel\, the\, -i\beta(\omega_0)\tilde{E}(z,\omega-\omega_0)\, terms$$

$$\frac{\partial}{\partial z}\tilde{E}(z,\omega-\omega_0) = -i\frac{d\beta}{d\omega}|_{\omega_0}(\omega-\omega_0)\tilde{E}(z,\omega-\omega_0) - \frac{i}{2}\frac{d^2\beta}{d\omega^2}|_{\omega_0}(\omega-\omega_0)^2\tilde{E}(z,\omega-\omega_0)$$

In order to convert this into an equation governing the pulse envelope in the time domain, we do Fourier transform:

First , the left-hand side becomes

$$\frac{1}{2\pi}\int_{-\infty}^{+\infty}\frac{\partial}{\partial z}\tilde{E}(z,\omega-\omega_0)e^{i\omega t}d\omega = \frac{1}{2\pi}\frac{\partial}{\partial z}\int_{-\infty}^{+\infty}\tilde{E}(z,\omega-\omega_0)e^{i\omega t}d\omega$$

$$= e^{i\omega_0 t}\cdot\frac{1}{2\pi}\frac{\partial}{\partial z}\int_{-\infty}^{+\infty}\tilde{E}(z,\omega-\omega_0)e^{i(\omega-\omega_0)t}d(\omega-\omega_0)$$

$$= e^{i\omega_0 t}\frac{\partial}{\partial z}\tilde{E}(z,t)$$

where we have used Eqn. (3).

Right-hand side:

$$\frac{1}{2\pi}\int_{-\infty}^{+\infty}\frac{d\beta}{d\omega}|_{\omega_0}(\omega-\omega_0)\tilde{E}(z,\omega-\omega_0)e^{i\omega t}d\omega$$

$$= \frac{d\beta}{d\omega}|_{\omega_0}e^{i\omega_0 t}\frac{1}{2\pi}\int_{-\infty}^{+\infty}(\omega-\omega_0)\tilde{E}(z,\omega-\omega_0)e^{i(\omega-\omega_0)t}d(\omega-\omega_0)$$

$$= -i\frac{d\beta}{d\omega}|_{\omega_0}e^{i\omega_0 t}\frac{1}{2\pi}\int_{-\infty}^{+\infty}\tilde{E}(z,\omega-\omega_0)\frac{\partial}{\partial t}e^{i(\omega-\omega_0)t}d(\omega-\omega_0)$$

$$= -i\frac{d\beta}{d\omega}|_{\omega_0} e^{i\omega_0 t}\frac{\partial}{\partial t}\{\frac{1}{2\pi}\int\limits_{-\infty}^{+\infty}\tilde{E}(z,\omega-\omega_0)e^{i(\omega-\omega_0)t}d(\omega-\omega_0)\}$$

$$= -i\frac{d\beta}{d\omega}|_{\omega_0} e^{i\omega_0 t}\frac{\partial}{\partial t}\tilde{E}(z,t)$$

In fact, it can generally be shown that

$$[i(\omega-\omega_0)]^n\tilde{E}(z,\omega-\omega_0) = \mathcal{F}\{\frac{\partial^n}{\partial t^n}\tilde{E}(z,t)\}$$

thus

$$\frac{1}{2\pi}\int\limits_{-\infty}^{+\infty}\frac{d^2\beta}{d\omega^2}|_{\omega_0}(\omega-\omega_0)^2\tilde{E}(z,\omega-\omega_0)e^{i\omega t}d\omega$$

$$= -e^{i\omega_0 t}\frac{d^2\beta}{d\omega^2}|_{\omega_0}\frac{\partial^2}{\partial t^2}\tilde{E}(z,t)$$

Substituting back into the wave eqn. and dividing out the $e^{i\omega_0 t}$ term, we find

$$\frac{\partial}{\partial z}\tilde{E}(z,t) = -\frac{d\beta}{d\omega}|_{\omega_0}\frac{\partial}{\partial t}\tilde{E}(z,t) + \frac{i}{2}\frac{d^2\beta}{d\omega^2}|_{\omega_0}\frac{\partial^2}{\partial t^2}\tilde{E}(z,t)$$

Recall we defined the group velocity as

$$\frac{1}{v_g} = \frac{\partial\beta}{\partial\omega}|_{\omega_0} = \beta'$$

which yields the '**parabolic equation**'

$$\frac{\partial}{\partial z}\tilde{E}(z,t) + \frac{1}{v_g}\frac{\partial}{\partial t}\tilde{E}(z,t) = \frac{i}{2}\frac{d^2\beta}{d\omega^2}|_{\omega_0}\frac{\partial^2}{\partial t^2}\tilde{E}(z,t)$$

(1) Note that if the group velocity dispersion is zero ($\beta'' = 0 \Rightarrow \beta' = \frac{1}{v_g} = constant$), then

$$\frac{\partial}{\partial z}\tilde{E}(z,t) + \frac{1}{v_g}\frac{\partial}{\partial t}\tilde{E}(z,t) = 0$$

This equation is satisfied for **any** function

$$\tilde{E}(z,t) = \tilde{E}(z - v_g t)$$

Thus any reasonable pulse (i.e. obeying the SVEA, and not so broadband that the approximation for β breaks down), not just the Gaussian pulse we treated earlier, will propagate with an unchanged envelope function when $\beta'' = 0$. Consequently, v_g = group velocity concept is valid for any 'reasonable' pulse, not just a Gaussian. To repeat, the envelope moves with the group velocity, which may be different from the phase velocity of the carrier wave.

(2) Note that the right-hand side of the parabolic eqn. is **imaginary,** so that it contributes to the **phase** of the pulse as it propagates. Thus if $\beta'' \neq 0$, the pulse envelope will be **distorted** with propagation. Later, when we discuss space-time analogies, we will see how you can think of this term as a kind of 'complex diffusion' responsible for pulse spreading, etc.

(3) A preview of things to come: terms can be added to the parabolic equation to describe additional modifications to the pulse as it propagates. For example, addition of a Kerr nonlinearity gives rise to the 'nonlinear Schrodinger eqn.', and addition of gain and absorptioin terms gives rise to a 'master eqn.' which describes the evolution of a pule in a passively mode-locked laser.

Before we go on to discuss dispersive pulse broadening and comopression, it will be useful to obtain some practical relations for the various dispersion parameters. The propagation constant $\beta(\omega)$ is usually obtained from the index of refraction, expressed most often by the Sellmeier eqn. i.e. given $n(\lambda)$, we need $\beta, \beta', \beta'', \cdots$

$$\frac{d\beta}{d\omega} = \frac{d}{d\omega}[\frac{n(\omega)\omega}{c}] = \frac{\omega}{c}\frac{dn}{d\omega} + \frac{n}{c}$$

$$\frac{d^2\beta}{d\omega^2} = \frac{\omega}{c}\frac{d^2n}{d\omega^2} + \frac{2}{c}\frac{dn}{d\omega}$$

From Sellmeier's eqn., we can get $\frac{dn}{d\lambda}, \frac{d^2n}{d\lambda^2}$, etc. We need to relate $\frac{d}{d\lambda}$ to $\frac{d}{d\omega}$: $\frac{d}{d\omega} = \frac{d\lambda}{d\omega}\frac{d}{d\lambda}$ (Chain's rule)

$$\lambda = \frac{2\pi c}{\omega}$$

$$\frac{d\lambda}{d\omega} = -\frac{2\pi c}{\omega^2} = -\frac{2\pi c}{\frac{4\pi^2 c^2}{\lambda^2}} = -\frac{\lambda^2}{2\pi c}$$

(1) $\frac{d}{d\omega} = -\frac{\lambda^2}{2\pi c}\frac{d}{d\lambda}$

similarly, one can derive

(2) $\frac{d^2}{d\omega^2} = \frac{\lambda^2}{(2\pi c)^2}[\lambda^2\frac{d^2}{d\lambda^2} + 2\lambda\frac{d}{d\lambda}]$

(3) $\frac{d^3}{d\omega^3} = -\frac{\lambda^3}{(2\pi c)^3}[\lambda^3\frac{d^3}{d\lambda^3} + 6\lambda^2\frac{d^2}{d\lambda^2} + 6\lambda\frac{d}{d\lambda}]$

Plugging these in the expressions for $\beta, \beta', \beta'', \cdots$ one finds

(4) $\frac{d\beta}{d\omega} = \frac{\omega}{c}\frac{dn}{d\omega} + \frac{n}{c} = \frac{1}{c}(n - \lambda\frac{dn}{d\lambda})$

(5) $\frac{d^2\beta}{d\omega^2} = \frac{\omega}{c}\frac{d^2n}{d\omega^2} + \frac{2}{c}\frac{dn}{d\omega} = \frac{\lambda}{(2\pi c)}\frac{\lambda^2}{c}\frac{d^2n}{d\lambda^2}$

(6) $\frac{d^3\beta}{d\omega^3} = \frac{3}{c}\frac{d^2n}{d\omega^2} + \frac{\omega}{c}\frac{d^3n}{d\omega^3} = -\frac{\lambda^2}{(2\pi c)^2}\frac{1}{c}(3\lambda^2\frac{d^2n}{d\lambda^2} + \lambda^3\frac{d^3n}{d\lambda^3})$

The point to remember: it is the curvature $\frac{d^2n}{d\lambda^2}$ that determines the GVD.

Also note that our dispersion parameter D $(ps/m \cdot nm)$

$$D = -\frac{1}{l}\frac{\Delta\tau_p}{\Delta\lambda}$$

$$= -\frac{d^2\beta}{d\omega^2}\frac{\omega}{\lambda}$$

$$= -\frac{\lambda}{(2\pi c)} \frac{\lambda^2}{c} \frac{d^2 n}{d\lambda^2} - \frac{2\pi c}{\lambda^2}$$

$$D = -\frac{\lambda}{c} \frac{d^2 n}{d\lambda^2}$$

example:

SF-10 flint glass, $\lambda_0 = 620$ nm.

Expansion of Sellmerer's eqn. yields

$$\frac{d^2 n}{d\lambda^2} \simeq 0.392 \, \mu m^{-2}$$

$$D = 8.1 \times 10^{-4} \frac{S}{m^2}$$

CPM ?? laser output: typically $\tau_p \simeq 60 \, fs$, $\Delta\lambda \simeq 8 \, nm$

In 1 mm flint, the pulse broadens by

$$\Delta\tau_p = DL\Delta\lambda = 6 \, fs \qquad (10\% \, broadening)$$

0.2 sign of GVD

We have seen that the dispersive pulse broadening is determined by

$$\beta'' = \frac{d}{d\omega}(\frac{1}{v_g(\omega)}) = -\frac{1}{v_g^2(\omega)} \frac{dv_g(\omega)}{d\omega}$$

$$= \frac{\lambda}{2\pi c} \frac{\lambda^2}{c} \frac{d^2 n}{d\lambda^2}$$

For propagation in **transparent dielectrics** in the **optical region** of the spectrum, we found from the Lorentz model (Sellmeier eqn.)

You can confirm algebraically what is apparent graphically: the index of refraction decreases with wavelength (i.e. increases with frequency), so that red travels faster than blue. In fact

$$\frac{d^2n}{d\lambda^2} > 0$$

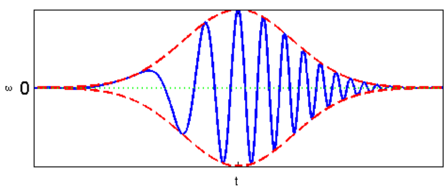
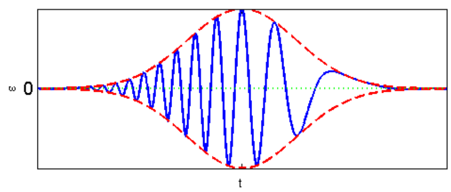
i.e. $n(\lambda)$ is concave upward, which is called **normal,** or more precisely, **'positive dispersion'.** $\Leftrightarrow \beta'' > 0$

In the regions where there is strong absorption (i.e. on resonance), the sign of the dispersion is opposite:

$$\text{on resonance} \leftrightarrow \text{'anomalous dispersion'} = \text{'negative dispersion'}$$

The terminology of normal and anomalous dispersion is a historical artifact, and conveys no physical sense of what is going on, so we will try to avoid it. The important thing to remember is

Table 1: normal and anomalous dispersion

positive	negative
$\beta'' > 0$	$\beta'' < 0$
$\frac{dv_g}{d\omega} < 0$	$\frac{dv_g}{d\omega} > 0$
red faster	blue faster
positive chirp	negative chirp
	

Clearly, if an optical pulse is always propagating through dielectric media, it will continually acquire more positive chirp, and hence it will just get longer and longer with propagation. We need to find some optical systems with **negative** GVD (**without** dispersion!), so we can **compensate** the positive chirp and maintain short pulses.

0.3 Notation

Before we go on to consider some systems with negative dispersion, we are going to shift our notation slightly from that of Siegman, both for convenience and to use the standard notation in the correct ultrafast optics literature. We have shown that the phase part of the transfer function is

$$e^{-i\Phi(\omega)} = e^{-i\beta(\omega)z}$$

for propagation in the z direction. We have looked at pulse propagation by expanding $\beta(\omega)$. However, all that really matters in the end is the **total phase** a pulse acquires on propagation

$$\Phi(\omega) = \beta(\omega)z$$

where z is the physical path length. Thus instead of considering β, many authors consider the phase

$$\Phi(\omega) = \Phi(\omega_0) + \Phi'(\omega_0)(\omega-\omega_0) + \frac{1}{2}\Phi''(\omega_0)(\omega-\omega_0)^2$$

$$+\frac{1}{6}\Phi'''(\omega_0)(\omega-\omega_0)^3 + \frac{1}{24}\Phi^{(4)}(\omega_0)(\omega-\omega_0)^4 + \cdots$$

where $\Phi''(\omega_0) = 2^{nd}$-order dispersion $= \beta''z$, $\Phi'''(\omega_0) = 3^{rd}$-order dispersion $= \beta'''z$, etc.

Note that the expansion terms are just z times the expressions for the derivations of β.

e.g.

$$\Phi' = \frac{d\beta}{d\omega}z = \frac{z}{c}(n - \lambda\frac{dn}{d\lambda})$$

$$\Phi'' = \frac{d^2\beta}{d\omega^2}z = \frac{\lambda z}{(2\pi c)}\frac{\lambda^2}{c}\frac{d^2n}{d\lambda^2}$$

Recall that we showed explicitly for a Gaussian pulse that the envelope was delayed in time by

$$t_g = \frac{z}{v_g(\omega_0)} = \beta' z$$

Thus we have a nice physical interpretation of the expansion coefficients in the expansion of $\Phi(\omega)$:

$\Phi'(\omega_0) = \frac{d\Phi}{d\omega}|_{\omega_0} =$ ' group delay' (in fs)

$\Phi''(\omega_0) = \frac{d\Phi}{d\omega}|_{\omega_0} =$ ' group delay dispersion' (GDD, in fs^2)

Of course, the interpretation of Φ' as the group delay is not specific to Gaussian pulses:

(1) the parabolic equation showes that $v_g = \frac{z}{\Phi'}$ describes the envelope motion for any reasonable envelope funciton.

(2) on homework, you will show directly by Fourier transforming

$$E(\omega - \omega_0)e^{i\Phi'(\omega_0)(\omega-\omega_0)}$$

that the effect of Φ' is just to shift the time origin.

Now that we konw that $\Phi'(\omega_0)$ is the group delay d, the pulse envelope after propagating a distance z, we can see what the physical effect of higher order phase terms is.

$$\Phi(\omega) = \Phi(\omega_0) + \Phi'(\omega_0)(\omega - \omega_0) + \frac{1}{2}\Phi''(\omega_0)(\omega - \omega_0)^2 + \frac{1}{6}\Phi'''(\omega_0)(\omega - \omega_0)^3$$

$\Phi'(\omega_0)(\omega - \omega_0)$: group delay; $\frac{1}{2}\Phi''(\omega_0)(\omega - \omega_0)^2$: GDD

Since the group delay of the peak of the pulse is given by $\Phi'(\omega_0)$, we can ask, what is the group delay at an arbitrary frequency ω?

$$t_g(\omega) = \Phi'(\omega) = \Phi'(\omega_0) + \Phi''(\omega_0)(\omega - \omega_0) + \frac{1}{3}\Phi'''(\omega_0)(\omega - \omega_0)^2$$

$\Phi'(\omega_0)$: group delay at carrier; $\Phi''(\omega_0)(\omega - \omega_0)$: group delay depends linearly on $(\omega - \omega_0)$ (Figure 1); $\frac{1}{3}\Phi'''(\omega_0)(\omega - \omega_0)^2$: group delay $\propto (\omega - \omega_0)^2$ (Figure 1).

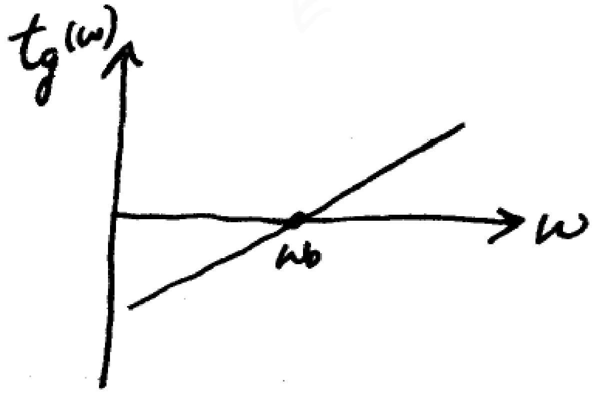


Figure 1: Linear chirp. Of course, it corresponds to a quadratic phase in frequency domain. Note: symmetric pulse stretching.

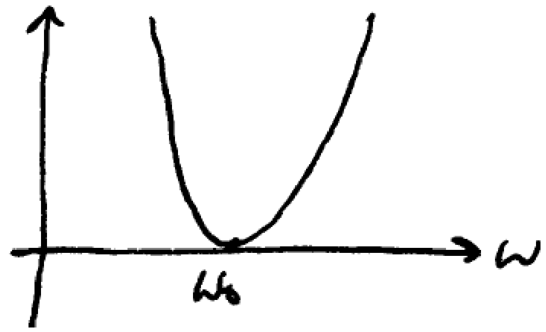


Figure 2: Quadratic chirp⟺Cubic phase delay⟺asymmetric pulse stretching. (both red and blue delayed)

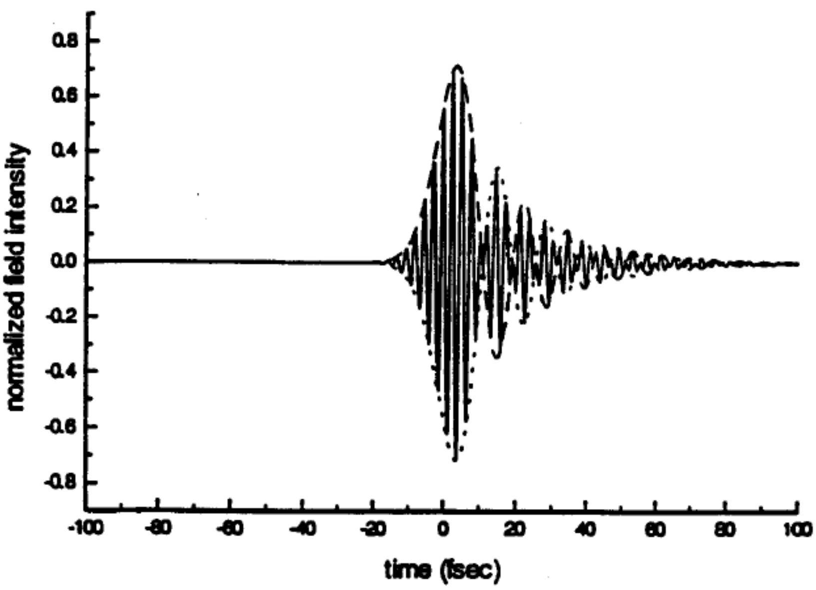


Figure 3: Gaussian pulse after acquiring $\Phi'''(\omega_0) = 200$ fs^3. The original pulse had $\Delta t = 5$ fs, $\omega_0 = 2\pi(0.375$ fs$^{-1})$, and $E_0 = 1$. The envelope function is included for clarity.