The second generalization of the notation is that if we are considering propagation in 2 or 3 dimension, we have to remember that $\beta(\omega)$ is a **vector** (the wavevector). Thus the phase part of the transfer function is really

$$e^{-i\beta z} \rightarrow e^{-i\vec{\beta}\cdot\vec{r}}$$

0.1 Grating dispersion[1, 2]

We can get the basic idea by considering the propagation of a beam between a parallel pair of transmission gratings(Figure 1)

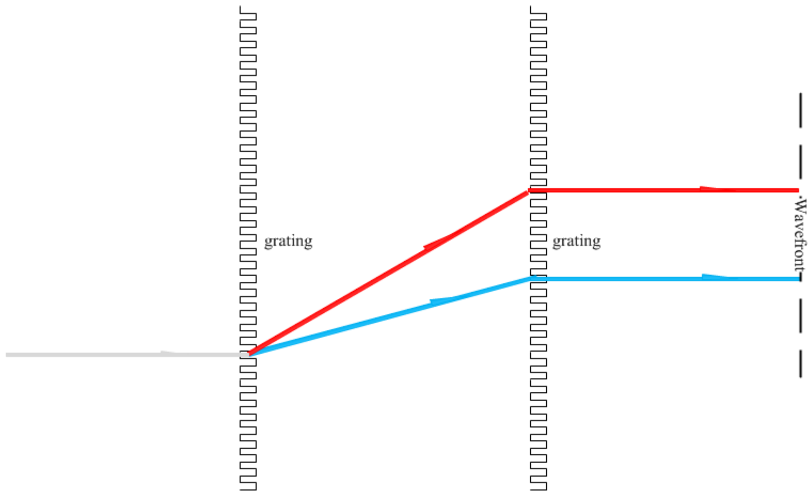

Figure 1: a beam between a parallel pair of transmission gratings

Note that the blue (short wavelength) components have a shorter optical path than the red components, so one might expect the GVD<0.

In the illustration, we have chosen the angle of incidence $\gamma = 0$, so the grating eqn. is

$$\sin\gamma - \sin\theta = m\frac{\lambda}{d}$$

or

$$-\sin\theta = m\lambda N$$

where $N = \frac{1}{d}$ = groove density.

1. Parallel grating $\Rightarrow$the output is **parallel** to the input for any λ.

2. diffraction angle larger for red than blue components⇒ optical path length longer for red.

3. the output has a 'spatial chirp' (red+blue components are separated spatially)

- is useful for some applications (we will see shortly)

- detrimental for others ⇒ solution: put a mirror (or root prism to change beam height) at the output to retroreflect the beam. After a **double pass**, there will be **no spatial chirp**, and the temporal **dispersion will be doubled** (i.e. pick up twice the optical path difference)

- note spatial chirp is noticeable for small diameters.

The basic problem is to find the path length vs. ω. Geometry for reflection gratings(Figure 2): We are after the distance $\overline{PABC}$. (P and C are chosen arbitrarily, but after all we are only offer **changes** in the distance with ω, so we are free to choose for convenience.)

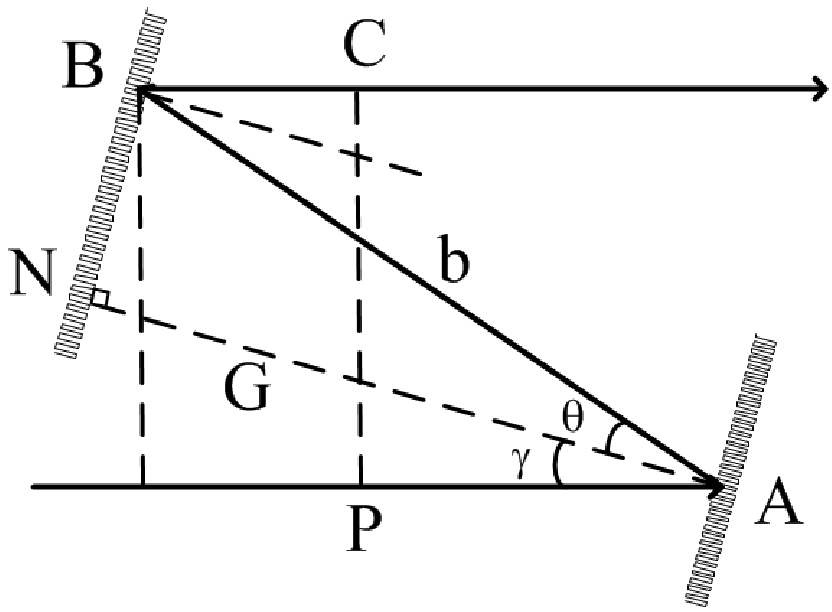

Figure 2: Geometry for reflection gratings.

From the figure

$$\overline{PABC} = b + b\cos(\gamma + \theta)$$

$$b = \frac{G}{\cos\theta}$$

$$\overline{PABC} = \frac{G}{\cos\theta}[1 + \cos(\gamma + \theta)] \equiv P$$

Grating eqn. for 1st order:

$$\sin\gamma - \sin\theta = \lambda N$$

Total phase shift on $PABC$ is

$$\varphi = \frac{\omega}{c} \times \overline{PABC} + phase\, shift\, on\, gratings$$

$$= \frac{\omega}{c} p + C(\omega)$$

We are interested in **relative** phases, so we can count from the most convenient point N, where the normal intersects with the grating.

Recall there is a relative phase shift of 2π at each groove of a grating $\Rightarrow$

$$C(\omega) = 2\pi \times \text{Number of grooves between } B \text{ and } N$$

$$= 2\pi N \times \overline{BN}$$

From figure, $\overline{BN} = G\tan\theta$

$$C(\omega) = 2\pi N G \tan\theta(\omega)$$

(+ phase shifts on reflection at surfaces, assuming independence on frequency$\Rightarrow$negligible)

Group delay:

$$\frac{d\varphi}{d\omega} = \frac{d}{d\omega}[\frac{\omega}{c}p + C(\omega)] = \frac{p}{c} + [\frac{\omega}{c}\frac{dp}{d\omega} + \frac{dC(\omega)}{d\omega}]$$

consider first the term in brackets

$$\frac{dC(\omega)}{d\omega} = 2\pi NG\frac{d(\tan\theta)}{d\theta}\frac{d\theta}{d\omega}$$

$$\frac{d(\tan\theta)}{d\omega} = \frac{1}{\cos^2\theta}\frac{d\theta}{d\omega}$$

$$\frac{dp}{d\omega} = \frac{dp}{d\theta}\frac{d\theta}{d\omega}$$

$$\frac{dp}{d\theta} = \frac{d}{d\theta}\{\frac{G}{\cos\theta}[1 + \cos(\gamma+\theta)]\}$$

$$= G\frac{d}{d\theta}\{\frac{1}{\cos\theta}[1 + \cos\gamma\cos\theta - \sin\gamma\sin\theta]\}$$

$$= G\frac{d}{d\theta}\{\frac{1}{\cos\theta} + \cos\gamma - \sin\gamma\tan\theta\}$$

$$= G\{\frac{\sin\theta}{\cos^2\theta} - \frac{\sin\gamma}{\cos^2\theta}\}$$

$= -G\frac{\lambda N}{\cos^2\theta}$ (from the grating equation)

$$\frac{\omega}{c}\frac{dp}{d\omega} + \frac{dC(\omega)}{d\omega} = [\frac{2\pi}{\lambda}(-G\frac{\lambda N}{\cos^2\theta}) + \frac{2\pi GN}{\cos^2\theta}]\frac{d\theta}{d\omega} = 0$$

$\Rightarrow$

$$\frac{d\varphi}{d\omega} = \frac{p}{c}$$

p is total pathlength.

The question naturally arises: is this cancellation just fortuitous, or is it **generally** true, so that $\frac{d\varphi}{d\omega} \equiv \frac{p}{c}$?

This question has been dealt with by Brorson and Haus.[2] They show that this result naturally follows as a direct consequence of **Fermat's Principle.**

Recall (e.g. from Born +Wolf sec. 3.3) that light travels along 'external paths' (i.e. a path such that the path length is stationary against small changes) with a velocity c.

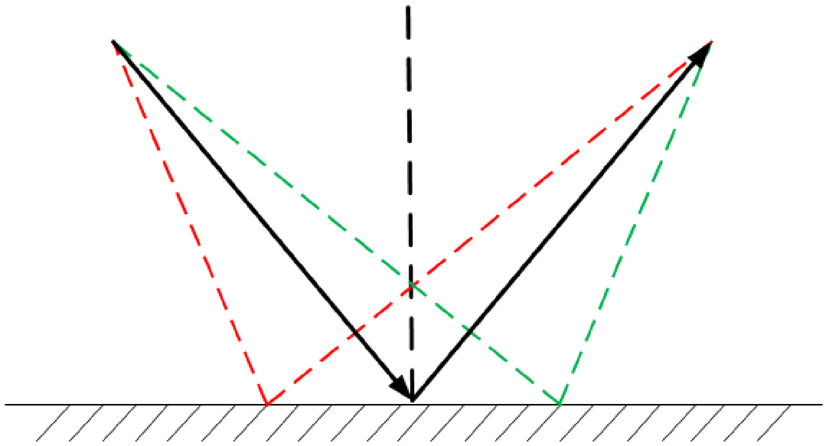

Figure 3: Actual light path (solid).

Thus if the path length is $p(\omega)$, the group delay must be

$$\tau(\omega) = \frac{p(\omega)}{c}$$

(assuming the medium is nondispersive, as it usually is since the gratings are in air or even vacuum.)

Why is the correction $C(\omega)$ needed? The argument was given by Treacy (who called it $R(\omega)$ - a better notation).

Consider the diffraction of a plane wave from a grating.(Figure 4)

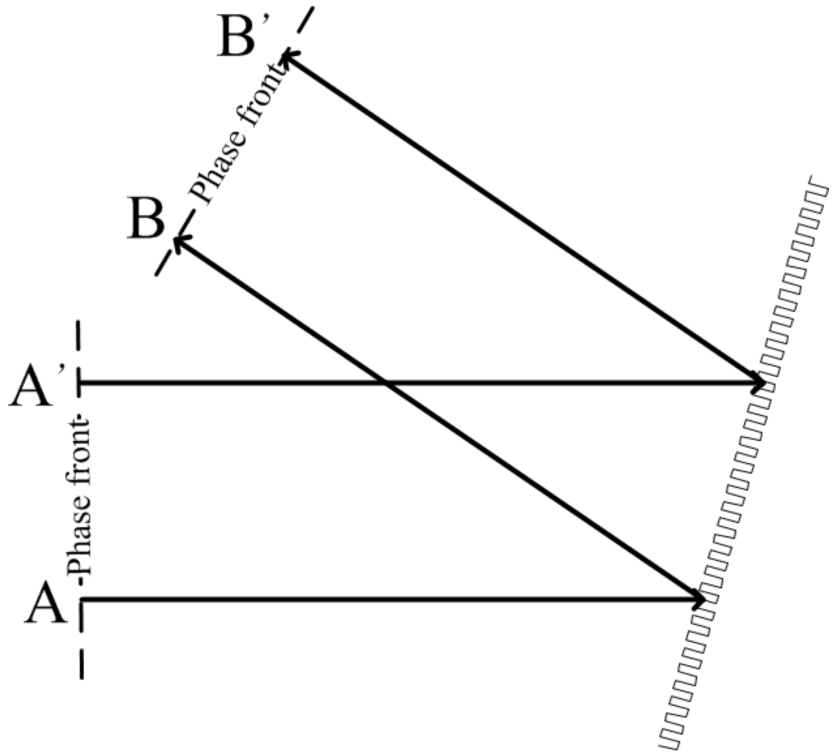

Figure 4: diffraction of a plane wave from a grating.

- clearly, $\overline{AB} < \overline{A'B'}$
- of course, the difference in optical paths is an integral multiple of wavelength.

The way to define a unique phase for the wavefront BB' is to note that the reason that the OPD is interger$\times\lambda$ is that each groove along the grating contributes a -2π phase shift (can be thought of as phase matching).

This phase shift along the grating surface is important for calculating the **variation** in phase with frequency, $\frac{d\phi}{d\omega}$.(Figure 5)

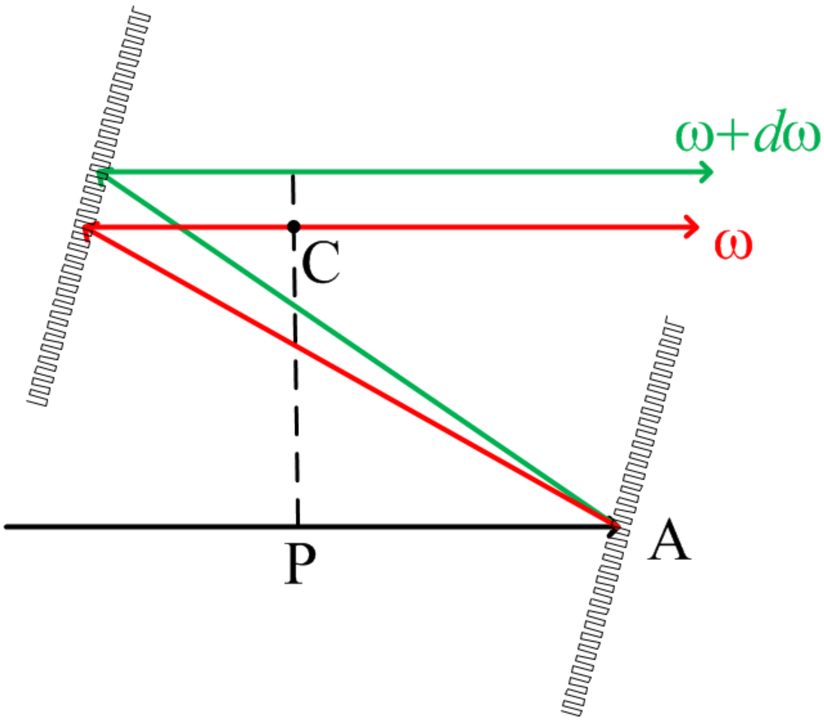

Figure 5: variation in phase with frequency.

Note that for a grating pair (Figure 6), $\overline{PC} = \overline{P'C'} \Rightarrow CC'$ is automatically a wavefront.

However, in measuring the phase from point P to the phase defined by C, $R(\omega)$ must be included in order to get a consistent result for $\frac{d\phi}{d\omega}$.

We are after the group delay dispersion (GDD)

$$\varphi'' = \frac{d^2\varphi}{d\omega^2} = \frac{1}{c}\frac{dp}{d\omega} = \frac{1}{c}\frac{dp}{d\theta}\frac{d\theta}{d\omega}$$

We now need $\frac{d\theta}{d\omega}$:

$$\sin\gamma - \sin\theta = \lambda N = \frac{2\pi c N}{\omega}$$

$$\frac{d}{d\omega}(\sin\gamma - \sin\theta) = -\cos\theta\frac{d\theta}{d\omega} = -\frac{2\pi c N}{\omega^2}$$

$$\frac{d\theta}{d\omega} = \frac{2\pi cN}{\omega^2 \cos\theta}$$

$$\varphi'' = \frac{1}{c}(\frac{-G\lambda N}{\cos^2\theta})(\frac{2\pi cN}{\omega^2 \cos\theta})$$

$$= -\frac{2\pi G\lambda N^2}{\omega^2 \cos^3\theta} = -\frac{G\lambda^3 N^2}{2\pi c^2 \cos^3\theta}$$

For the **double-pass** geometry, this is multiplied by 2:

$$\varphi'' = -\frac{G\lambda^3 N^2}{\pi c^2 \cos^3\theta}$$

This equation was first derived by Treacy[1], where it was expressed in terms of the incidence angle and $d = \frac{1}{N}$:

$$\varphi'' = -\frac{G\lambda}{\pi c^2} \frac{(\frac{\lambda}{d})^2}{[1 - (\sin\gamma - \frac{\lambda}{d})^2]^{\frac{3}{2}}}$$

The important thing about this equation is that the **group delay dispersion is always negative** for the parallel grating pair (θ is always between $\pm\frac{\pi}{2}$). Thus the grating pair can be used to **compensate** for th positie dispersion that stretches pulses as they propagate through normally dispersive media.

0.2 Higher order dispersion

Clearly higher order terms in the phase expansion can be derived simply from successive diffrentiation of φ''. The algebra is tedious and un-illuminating, so we will just give some results:

$$\varphi''' = -\frac{3\lambda}{2\pi c}\varphi''(\omega_0)\{1 + \lambda N[\frac{\lambda N - \sin\gamma}{1 - (\lambda N - \sin\gamma)^2}]\}$$

note that a real angle of diffraction implies

$$(\lambda N - \sin\gamma)^2 < 1$$

or

$$-1 < \lambda N - \sin\gamma < 1$$

$$1 + \lambda N \cdot \frac{\lambda N - \sin\gamma}{1 - (\lambda N - \sin\gamma)^2} = \frac{1 - (\lambda N - \sin\gamma)^2 + N\lambda(\lambda N - \sin\gamma)}{1 - (\lambda N - \sin\gamma)^2}$$

1. numrator= $1 - (\lambda N)^2 + 2\lambda N \sin\gamma - \sin^2\gamma + (\lambda N)^2 - \lambda N \sin\gamma$

$= 1 + \lambda N \sin\gamma - \sin^2\gamma = 1 + \sin\gamma(\lambda N - \sin\gamma) > 0$ ($|\sin\gamma| < 1$, $|(\lambda N - \sin\gamma)| < 1$)

2. denominator always > 0 (obvious)

3. $\varphi'' < 0$ derived above

$\Rightarrow \varphi''' > 0$ Third order dispersion of grating pair is **positive.**

This is an important fact. You can use the negative φ'' of normal dielectrics, but the sign of φ''' is positive for **both** gratings **and** normal dielectrics, so the **pulse will have cubic error** (i.e. **not** transform-limited!)

For reference, the fourth order dispersion (FOD) of the grating pair is

$$\varphi^{(4)}(\omega_0) = \frac{(2\varphi''')^2}{3\varphi''} + \{\frac{\lambda^2 N}{2\pi c[1 - (\lambda N - \sin\gamma)^2]}\}^2 \varphi''$$

When must higher order dispersion be taken into account? Physically, the answer must be that **larger** will be more likely to have substantial phase errors. We can make this a little more quantitative by looking at the ratio of TOD to SOD (GDD) for a pulse of bandwidth $\Delta\omega$

$$R = |\frac{\frac{1}{6}\varphi'''\Delta\omega^3}{\frac{1}{2}\varphi''\Delta\omega^2}| = |\frac{\varphi'''\Delta\omega}{3\varphi''}|$$

$$= \frac{\Delta\omega}{\omega}\{1 + \lambda N[\frac{\lambda N - \sin\gamma}{1 - (\lambda N - \sin\gamma)^2}]\} \ \ using \ \frac{\lambda}{2\pi c} = \frac{1}{\omega}$$

0.3 Practical implementation

Either **reflection** or **transmission** gratings may be used. Most gratings are fabricated to give maximum diffraction efficiency in the Littrow configuration (where the beam diffracts back on itself so $\theta = -\gamma$) (Figure 6)

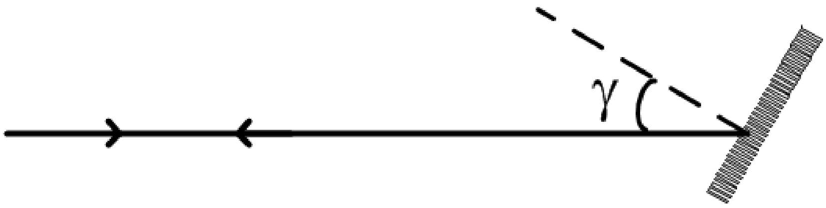

Figure 6: Littrow configuration.

Thus the grating compressor usually operates in the -1 order in the scheme of Figure 5. (Figure 7)

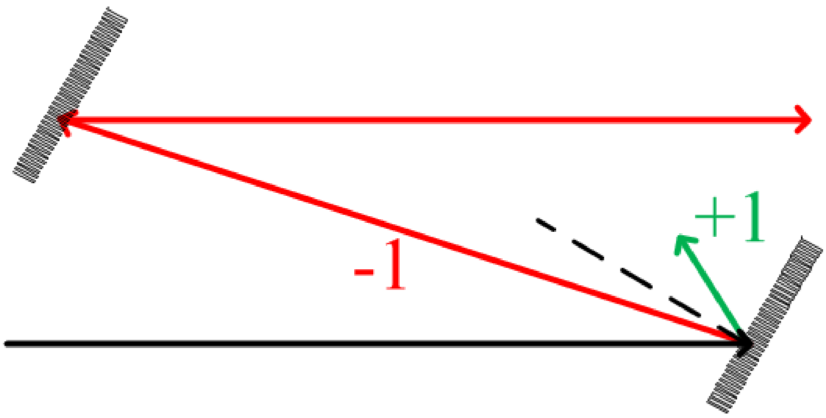

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Figure 7: The grating compressor usually operates in the -1 order.

Note:

1. if all the beams are confined to the plane, it is not possible to operate **exactly** at Littrow, frequently resulting in **reduced efficiency.**

2. you could tilt the gratings up or down, but that comes at the cost of other phase errors on the beam.

3. transmission gratings can be used exactly at Littrow.

4. note the difficulty of running near Littrow with nonzero beam diameters (fixed beam dia. $\Rightarrow$if you want low dispersion, you have to reduce the grating separation

⇒reduced efficiency. Going to too low a groove spacing N results in low efficiency since there are many diffracted orders)

5. why not always use transmission gratings?

- availability
- **bandwidth** is generally lower (i.e. over what frequency range $\Delta\omega$ is the diffraction efficiency high?)
- in high power systems: **damage**

For reasons of efficiency, you generally want higher-N gratings (lower-N gratings usually are not quite as efficient, especially if grating orders other than -1 are allowed.)

Why not arbitrarily high?

1. N needs to be low enough that the gratings can be far enough apart so the beams will not be chopped.
2. sensitivity to higher order phase errors.

Van ruald in his thesis has plotted $\frac{\varphi'''}{\varphi''}$ and $\frac{\varphi^{(4)}}{\varphi''}$ as a fraction of angle of incidence.(Figure 8) Note that

1. lower N ⇒ lower high order phase errors.
2. lower N ⇒ lower sensitivity to misalighment, i.e. variations in angle of incidencee
3. it is a log scale!

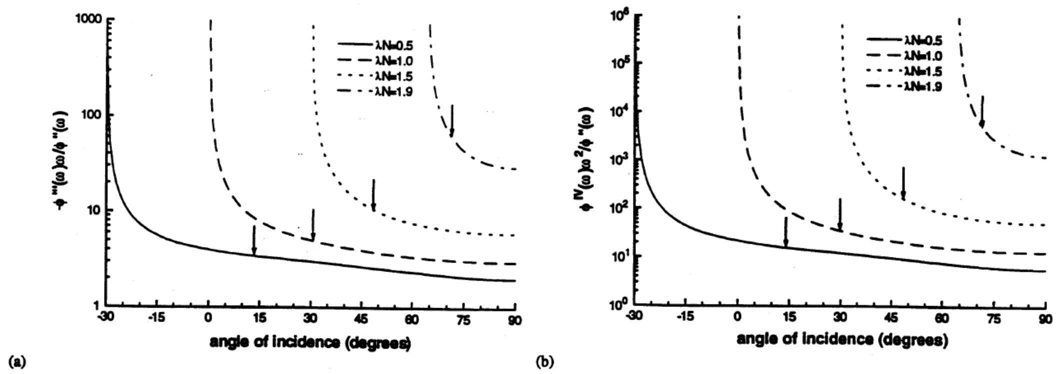

Figure 8: (a)$-\frac{\varphi'''(\omega_0)\omega}{\varphi''(\omega_0)}$, (b) $\frac{\varphi^{(4)}(\omega_0)\omega^2}{\varphi''(\omega_0)}$, vs. incident angle, γ, for a standard grating compressor made with various values of λN. Littrow incidence angle for each grating is noted by an arrow.

References

[1] E.B. Treacy, IEEE *J.Quant.Electron.* **QE. S**, 454 (1969)

[2] S.D. Brorson and H.A. Haus, *JOSA.* B S, 247 (1988)