Consider the **spatial analogy** to the pulse shown in Siegman Fig 9.5: the input pulse has a positive chirp (red in front, blue in back). By propagating through a negatively dispersive delay line, the pulse is compressed to its transform limit; there is no chirp on the input, and the envelope function (a sinc) is the Fourier transform of the input (a rect).

Spatial analogy:

Table 1: time-space analogy

time	space
rect	rect
positive chirp	converging beam (positive spatial frequencies $\leftrightarrow \theta_x < 0$ are at positive x)
sinc	sinc envelope at focus

Q: How do you get a rect beam with converging wavefront?

A: Use a lens (at finite apeture) to focus an incident plane wave.

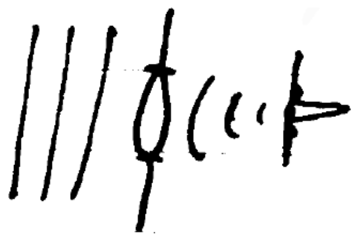

Figure 1: get a rect beam with converging wavefront.

$\Rightarrow$ we need to find a way to do something similar in the time domain, i.e. we need a **time lens.**

We can also consider the propagation problem from the point of view of a given source-plane field in the frequency domain:

$$E(\tau, \xi) = \frac{1}{\sqrt{\xi}} \int E_0(\tau_0, 0) e^{i(\tau-\tau_0)^2/2\beta''\xi} d\tau_0$$

$$= \frac{1}{\sqrt{\xi}} \int d\tau_0 e^{i(\tau-\tau_0)^2/2\beta''\xi} \int d\omega A(\omega, 0) e^{i\omega\tau_0}$$

$\int d\omega A(\omega,0)e^{i\omega\tau_0}$ is Fourier transform of input field.

reverse order of integration and carry out $\int d\tau_0$:

$$E(\tau,\xi) = \frac{1}{\sqrt{\xi}}\int d\omega A(\omega,0)\int d\tau_0 e^{i(\tau-\tau_0)^2/2\beta''\xi}e^{i\omega\tau_0}$$

$$= \frac{1}{\sqrt{\xi}}\int d\omega A(\omega,0)\{\sqrt{\frac{2\pi\beta''\xi}{i}}e^{i\xi\beta''\omega^2/2}e^{-i\omega\tau}\}$$

$$\propto \int d\omega A(\omega,0)e^{i\xi\beta''\omega^2/2}e^{-i\omega\tau}$$

(Kolner eqn. 25; apart from constants out front which we are ignoring)

The analogous expression for the diffraction problem is:

$$E(x,z) = \int E(\omega_0,0)e^{i\omega_k^2 z/2k}e^{-i\omega_x x}d\omega_x$$

where

$$\omega_x = -\frac{2\pi}{\lambda}\sin\theta_x \simeq -\frac{2\pi}{\lambda}\theta_x$$

is the 'spatial frequency'. (note we are following the notation we used in 537, where the not-necessarily-conventional minus sign is introduced to make the forms of the Fourier tranforms equavalent. To connect to Siegman's and Kolner's notation, we use $k_x = -\omega_x = \frac{2\pi}{\lambda}\theta_x$ and $S_x = \frac{\theta_x}{\lambda} = \frac{k_x}{2\pi}$ is refered to as the spatial frequency.)

Note that the physical picture at diffraction is that the higher spatial frequencies diffract away from the central (zero) spatial frequency, leading to a spatial frequency 'chirp'. (see Kolner Fig 4) In the time domain, the frequencies ?? the carrier 'diffract' away from the carrier.

In both cases, the 'diffraction' results from the multiplication of the input **spec-**

trum by a quadratic-frequency 'filter'; the output field (in time or real space) is the Fourier transform of the 'filtered' spectrum.

0.1 Space lenses and Time lenses

Now that we know that free space diffraction is analogous to dispersive pulse propagation, we might ask what the time-domain equivalent of a lens is

Consider a plane wave incident on a lens:

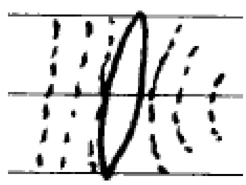

Figure 2: a plane wave incident on a lens

It is easy to see that the effect of a thin lens on the phase of the input wave can be described by a transfer function (see Goodman, *Fourier Optics*)

$$t_l(x,y) = e^{-ikn\Delta_0} e^{-i\frac{k}{2f}(x^2+y^2)}$$

where Δ_0 is max lens thickness, and f is focal length.

The overall phase shift can be ignored. The important thing is that a lens puts on a quadratic phase modulation in real space. (remember e^{ibt^2})

Note for future reference that a lens puts new spatial frequencies on the beam. A monochromatic plane wave has only one spatial frequency; after the lens, the spatial frequency spectrum is broadened by the lens, and the beam focuses or defocuses.

Following our space-time analogy, let us see what happens if we put a **quadratic phase modulation in time** on a pulse:

$$\phi(\tau) = -\frac{\omega_0}{2f_\tau}\tau^2$$

where we call f_τ the 'focal time'.

In general, we might suppose some physical process might induce a temporal phase change on a pulse, which we could approximate as

$$\varphi(\tau) \simeq \varphi(\tau_0) + \varphi'(\tau_0)(\tau\text{-}\tau_0)+\frac{1}{2}\varphi''(\tau_0)(\tau\text{-}\tau_0)^2$$

Then the effect of the second-order term will be to act as a time lens with

$$f_\tau = -\frac{\omega_0}{d^2\omega/d\tau^2}$$

Note that putting a quadratic temporal phase on the pulse **will change the pulse spectrum.** You will recall from homework #1 that ,while a quadratic frequency-domain phase could be performed with a linear optical system, the quadratic temporal phase introduced new spectral components, and this required some nonlinearity in the system(or at least a non-time-shift-invariant system).

The nonlinear optical effect yielding the most controlled quadratic phase modulation is the electro-optic effect. We will consider specifically an **electro-optic traveling-wave phase modulator.**

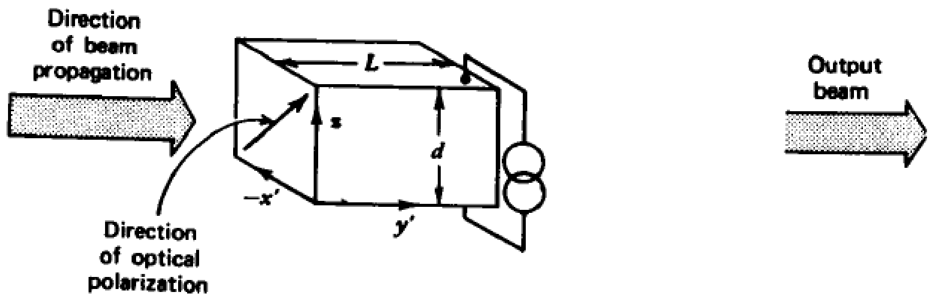

Figure 3: transverse electrooptic modulator using a KH_2PO_4 (KDP) crystal in which field is applied normal to the direction of propagation.

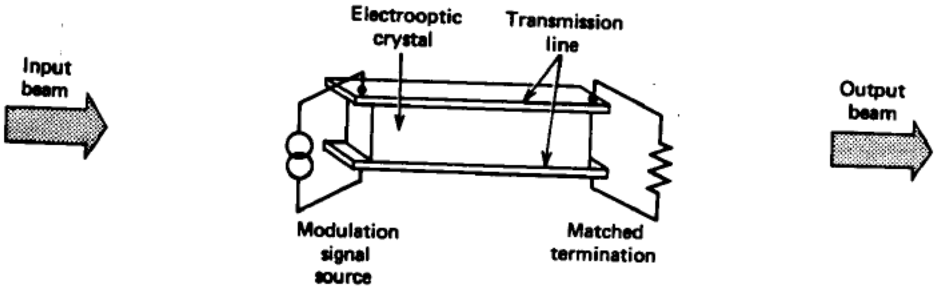

Figure 4: traveling wave electrooptic modulator.

We will assume that the optical pulse and modulation signal ($V_0 \cos \omega_m t$) are velocity-modulated, so the propagate together along the e-o crystal.

The index modulation induced by the microwave driving field is

$$\Delta n(z,t) = \Delta n_0 \cos(\omega_m t - k_m z)$$

(note: not a time-shift-invariant system!)

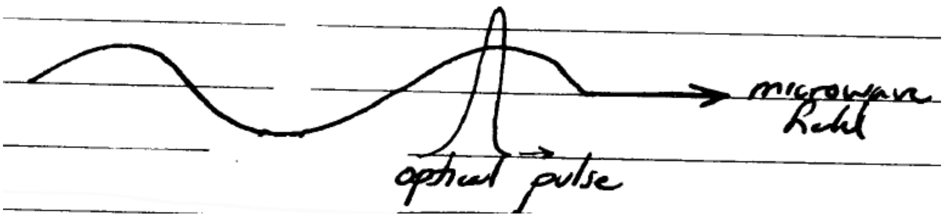

Figure 5: microwave modulation.

We can write the phase modulation at the crystal exit as

$$\Gamma(\xi, \tau) = \Gamma_0 \cos \omega_m \tau$$

where

$$\Gamma_0 = \frac{\omega_0 \Delta r_0}{c} \xi = \pi \frac{V}{V_\pi}$$

$$= \text{peak deviation}$$

V_π =half wave voltage

If the optical pulse is short compared to the microwave period, then

$$\Gamma(\xi,\tau) \simeq \Gamma_0[1 - \frac{(\omega_m\tau)^2}{2}]$$

$-\frac{(\omega_m\tau)^2}{2}$ indicates quadratic phase shift, as desired.

$$\frac{d^2\varphi}{d\tau^2} = \frac{d^2\Gamma}{d\tau^2} = -\Gamma_0\omega_m^2 \quad \Rightarrow f_\tau = \frac{\omega_0}{\Gamma_0\omega_m^2}$$

The transfer function of the time lens is therefore

$$H_l(\tau) = e^{-i\Gamma_0} e^{i\Gamma_0(\omega_m\tau)^2/2}$$

$$= e^{-i\Gamma_0} e^{i\omega_0\tau^2/2f_\tau}$$

A temporal version of the f-number may also be obtained: Recall $f^* = \frac{f}{D}$, where D =aperture size (diameter) $\Rightarrow$ need to extend the definition of aperture.

Clearly, there is no hard aperture for the cos-modulated time-lens. We can say, however, that the aperture is the **temporal window** in which the phase modulation is **predominantly quadratic.**

Kolner's definition:

$$\Gamma \simeq \Gamma_0[1 - \frac{(\omega_m\tau)^2}{2} + \frac{(\omega_m\tau)^4}{24}]$$

Within time window $|\tau| < \frac{\tau_a}{2}$, require the quartic term to be less than 2% of the quadratic term:

$$0.02\frac{(\omega_m\tau_a)^2}{2} \geqslant \frac{(\omega_m\tau_a)^4}{24}$$

$$\frac{0.24}{\omega_m^2} \geqslant (\frac{\tau_a}{2})^2 \qquad \Rightarrow \tau_a \approx \frac{1}{\omega_m}$$

$$\Rightarrow f_\tau^\# = \frac{f_\tau}{\tau_a} \simeq \frac{\omega_0/\Gamma_0\omega_m^2}{1/\omega_m} = \frac{\omega_0}{\Gamma_0\omega_m}$$

order-of magnitude numbers:

- $\omega_0 \sim 2 \times 10^{15} s^{-1}$
- $\omega_m \sim 2\pi \times 10^{10} s^{-1}$
- $\Gamma_0 \simeq 2\pi$
- $f_\tau = \frac{2\times10^{15}}{2\pi(2\pi\times10^{10})^2} \simeq 95\text{ns}$
- $f_\tau = \frac{2\times10^{15}}{2\pi(2\pi\times10^{10})} \simeq 5000\text{s}$ (a very slow lens!)

0.2 Aberrations

Recall from physical optics that ideal imaging is obtained as long as the phase is **quadratic** in the spatial variables. In our temporal imaging system, aberrations are introduced if the **quartic** phase term becomes significant.

The analogy in spatial imaging is **spherical aberration.**

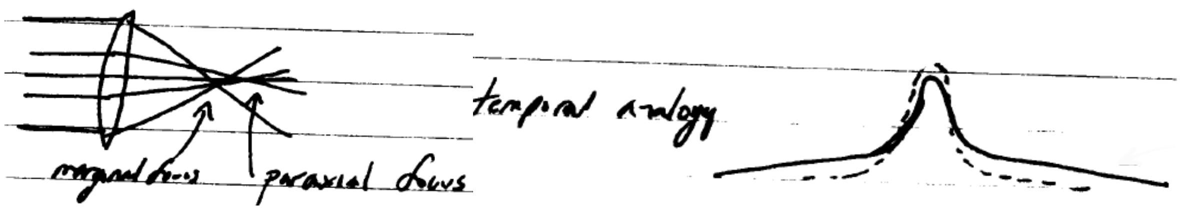

Figure 6: spherical aberration and its temporal analogy.

0.3 Temporal Imaging

Finally, we would like to see how to extend the analogy to **imaging.** In the spatial domain, we have the familiar picture

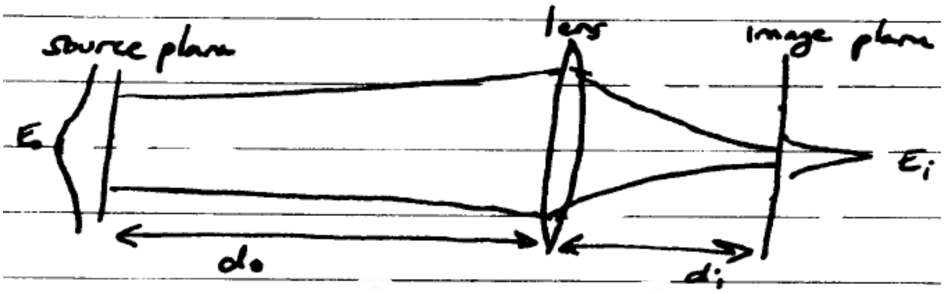

Figure 7: free space propagation. (diffraction)

The imaging condition is

$$\frac{1}{d_0} + \frac{1}{d_i} = \frac{1}{f}$$

and the magnification is $M = -\frac{d_i}{d_0}$ (the minus sign indicating inversion of the image when both d_i and d_0 are positive and f is positive)

Why bother? If it is possible to construct the analogous imaging system in the time domain, we could immediately imagine its applications:

1. pulse compressor (start with long pulse; end up with a short one)

2. temporal microscope (image a short pulse into a long one, so it is easier to measure)

The analogous optical arrangements for spatial and temporal imaging are shown in Kolner Figs. 6&7

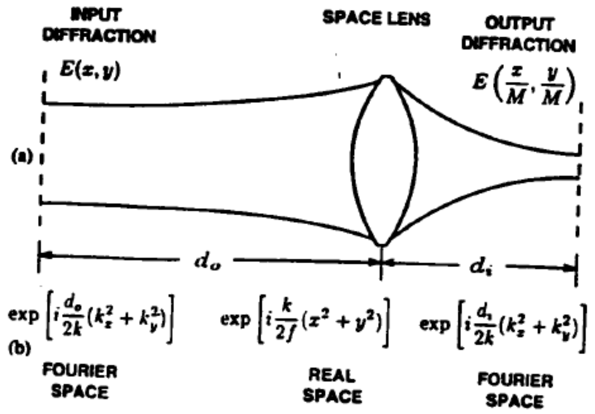

Figure 8: (a) Configuration for conventional spatial imaging. Output field envelope $E(\frac{y}{M}, \frac{x}{M})$ is a magnified version of the input field envelope $E(x, y)$ where $M = -\frac{d_i}{d_0}$. (b) Analysis is carried out by cascading the three processes: input diffraction [quadratic phase transformation in Fourier-space variables (k_x, k_y)] $\Rightarrow$ lens [quadratic phase transformation in real-space variables(x,y)] $\Rightarrow$ output diffraction.

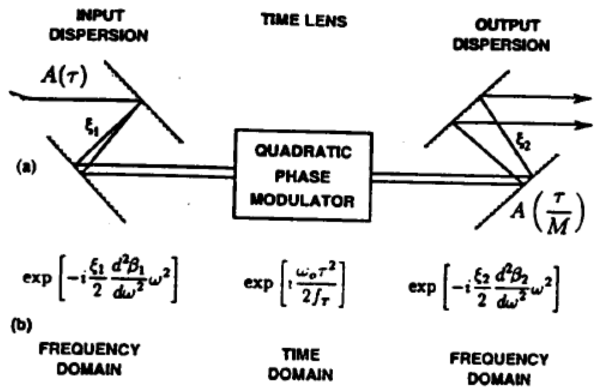

Figure 9: (a) Temporal imaging configuration. Input and output dispersions (shown here as diffraction-grating pairs) play the role of free-space diffraction while a quadratic phase modulator acts as a lens in the time domain. Output waveform envelope $A(\tau/M)$ is a magnified version of the input envelope $A(\tau)$, where $M = -(\xi_2 d^2\beta_2/d\omega^2)/(\xi_1 d^2\beta_1/d\omega^2)$. (b) Analysis is carried out by cascading the three processes: input dispersion [quadratic phase transformation in frequency domain (ω)] $\rightarrow$ time lens [quadratic phase modulation in time (τ)] $\rightarrow$ output dispersion. Compare with the spatial analog shown in Fig 8.